

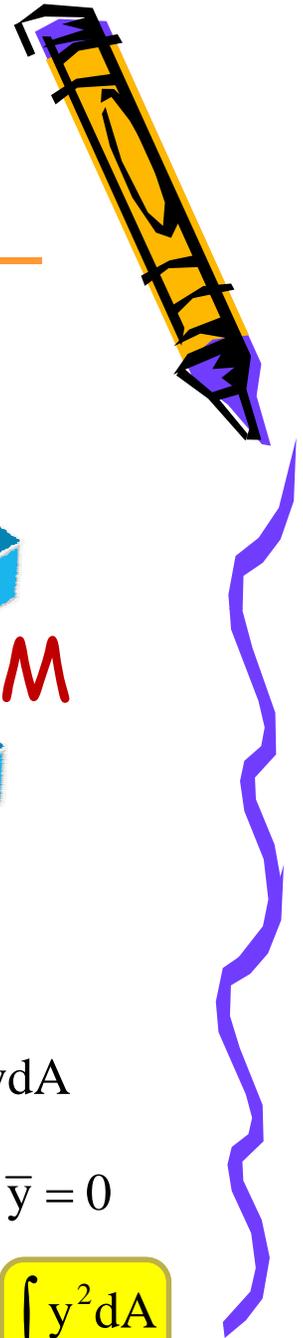
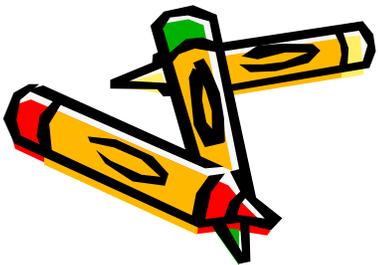
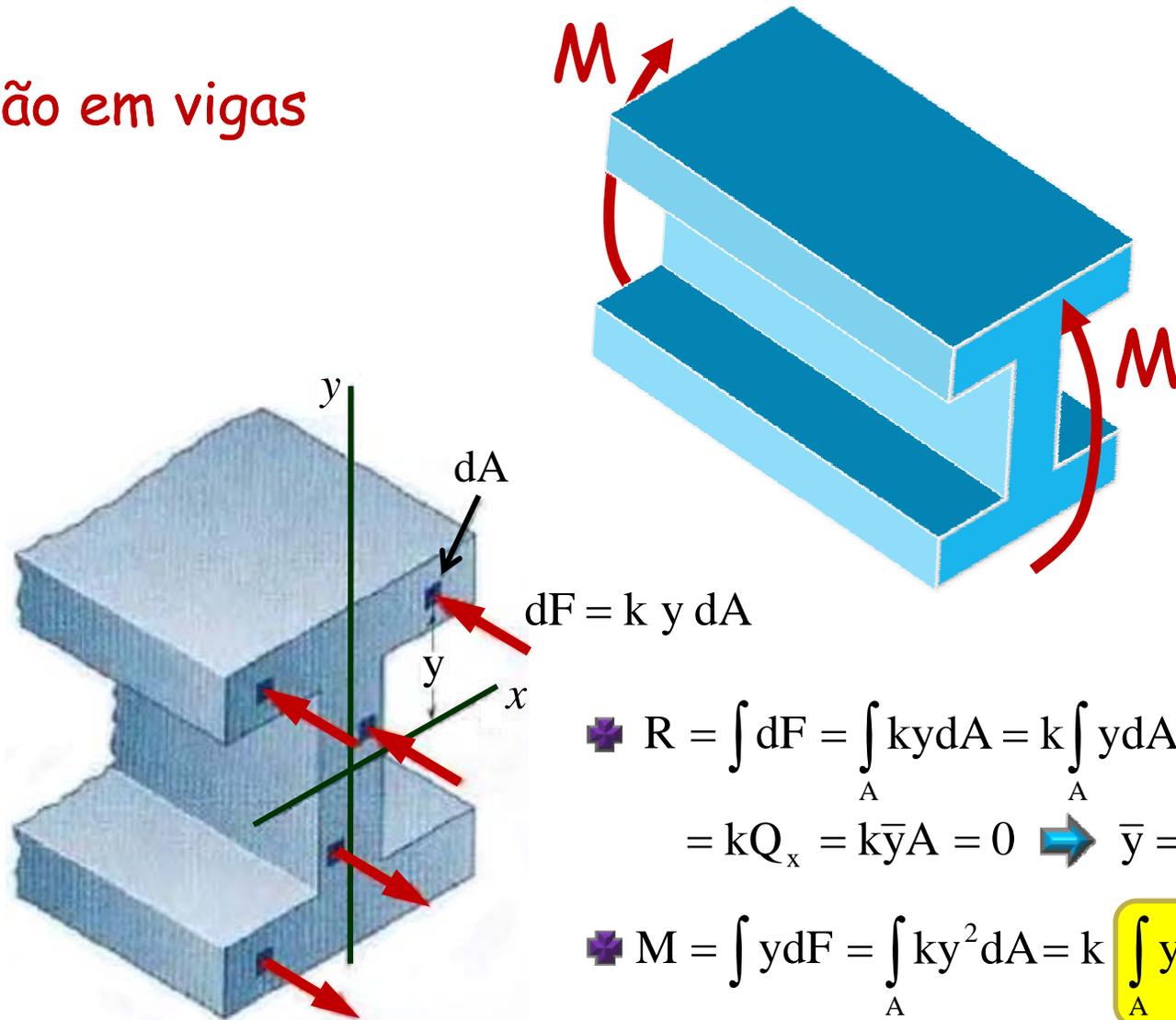
Disciplina: Mecânica dos Sólidos 1
Código: ECIV018
Professor: Eduardo Nobre Lages

Forças Distribuídas: Momentos de Inércia



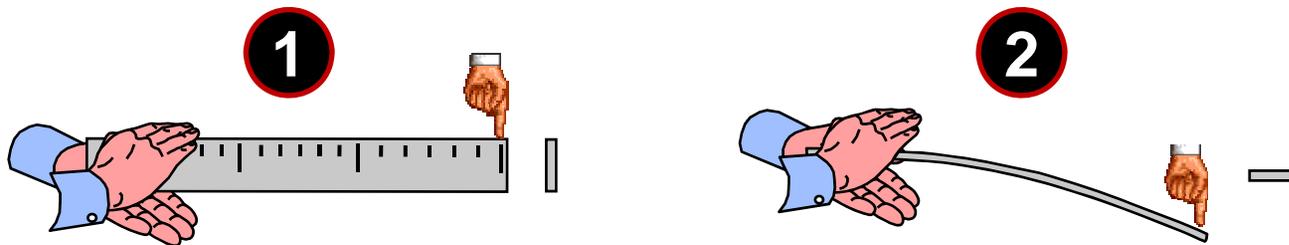
Motivação

● Flexão em vigas



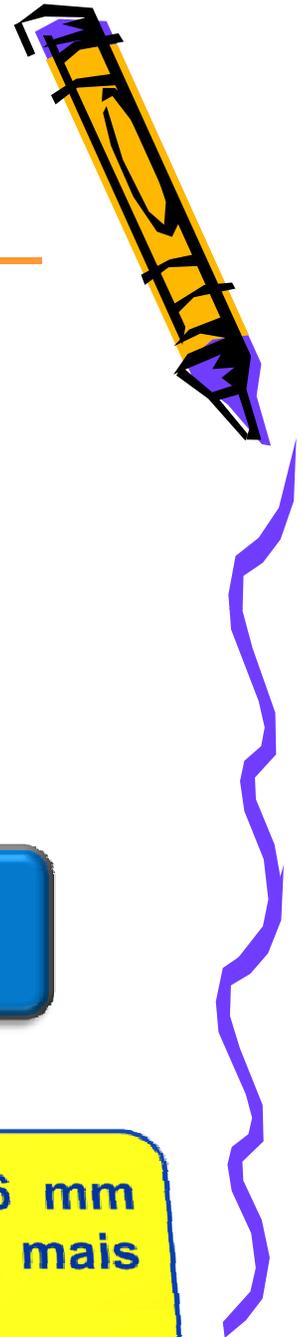
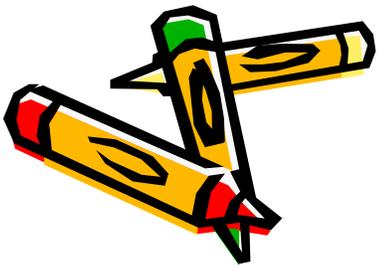
Motivação

● Flexão em vigas



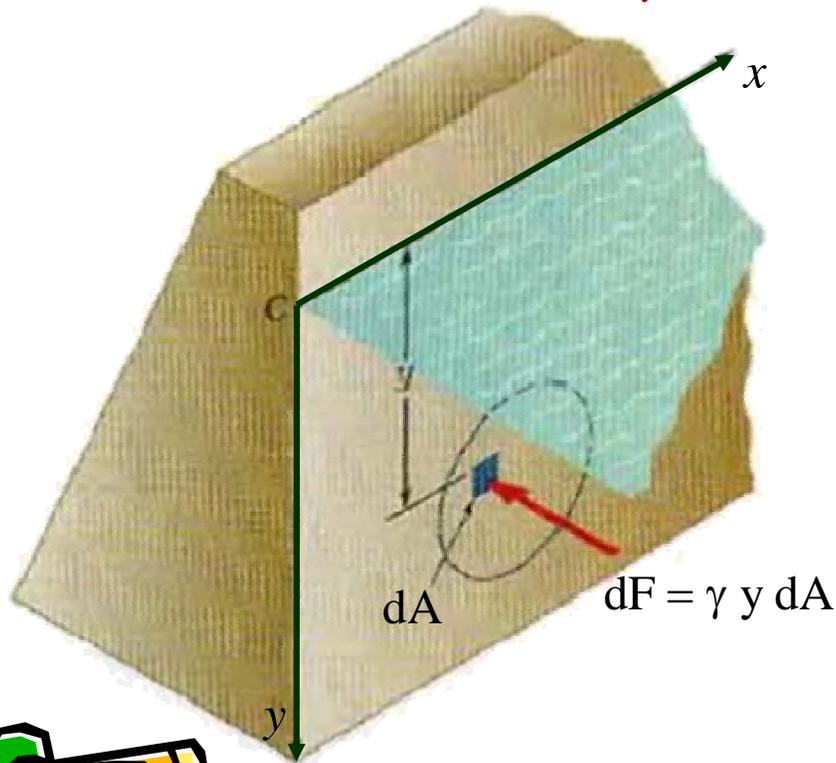
Consumindo-se um mesmo volume de material, é possível modificar a rigidez à flexão da estrutura.

Exemplo: Para uma seção de 2 mm x 36 mm (seção retangular), o arranjo 1 é 324 vezes mais rígido que o arranjo 2.



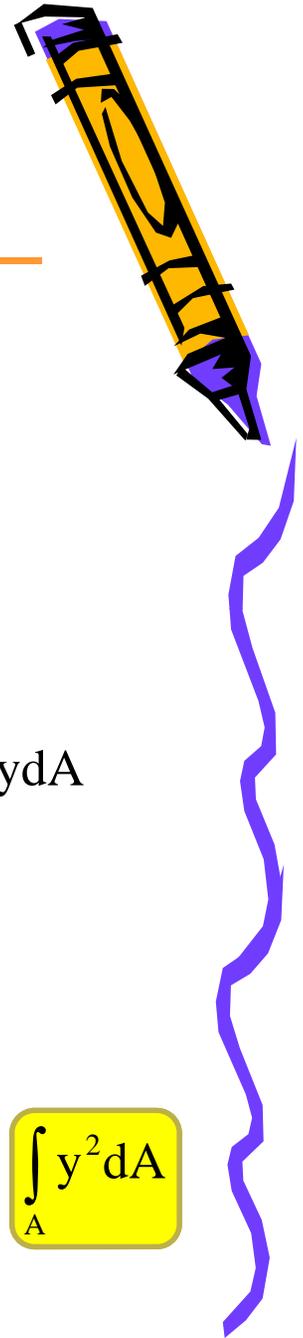
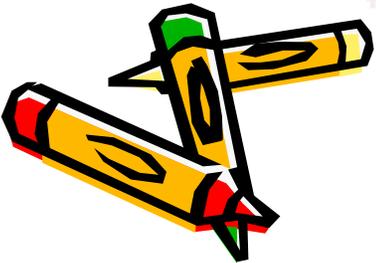
Motivação

● Pressão sobre comportas

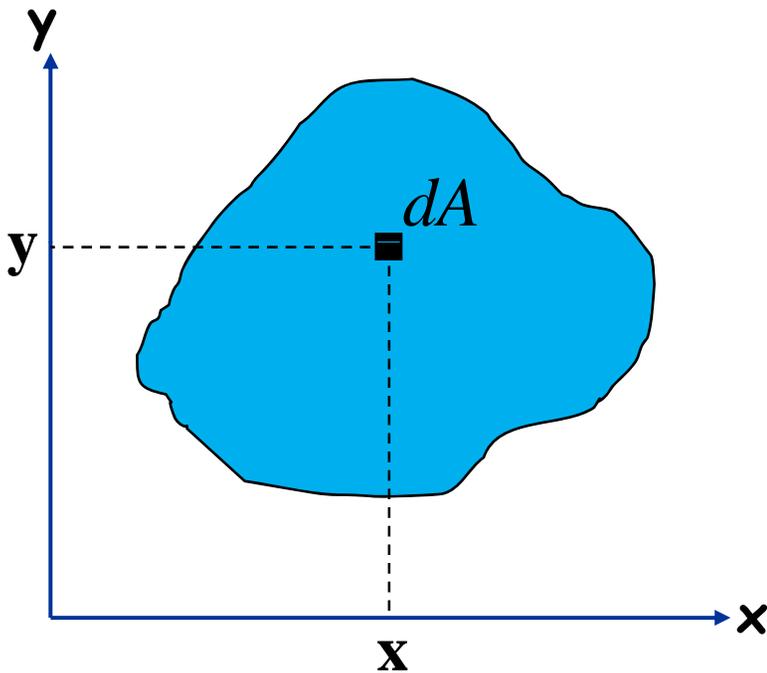


$$\begin{aligned} \text{✦ } R &= \int dF = \int_A \gamma y dA = \gamma \int_A y dA \\ &= \gamma Q_x = \gamma \bar{y} A \end{aligned}$$

$$\text{✦ } M = \int y dF = \int_A \gamma y^2 dA = \gamma \int_A y^2 dA$$



Momento de Inércia ou Momento de 2ª Ordem

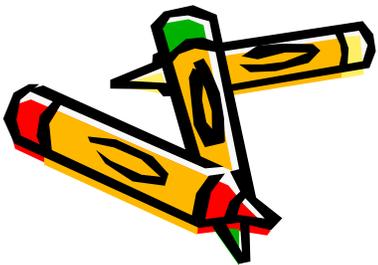


Momento de inércia ou de 2ª ordem em relação ao eixo x

$$I_x = \int_A y^2 dA$$

Momento de inércia ou de 2ª ordem em relação ao eixo y

$$I_y = \int_A x^2 dA$$



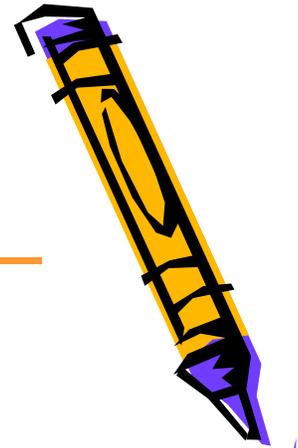
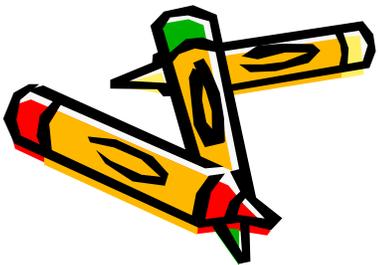
Determinação dos Momentos de Inércia por Integração

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

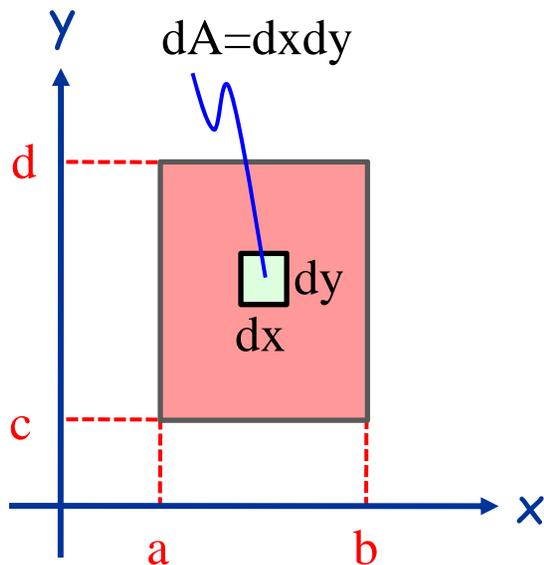
- Em princípio, para quantificação dos momentos de 2ª ordem (ou momentos de inércia), esses são calculados a partir de integrais duplas no domínio representativo da região estudada, onde se deve escrever o elemento infinitesimal de área dA de acordo com a conveniência das coordenadas

de descrição da região tratada.

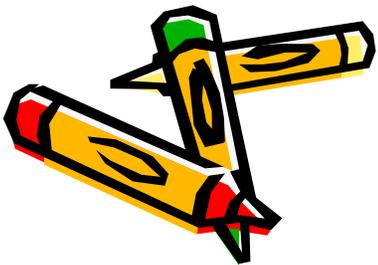


Determinação dos Momentos de Inércia por Integração Dupla

$$D = \{ (x, y) \mid a \leq x \leq b \text{ e } c \leq y \leq d \}$$

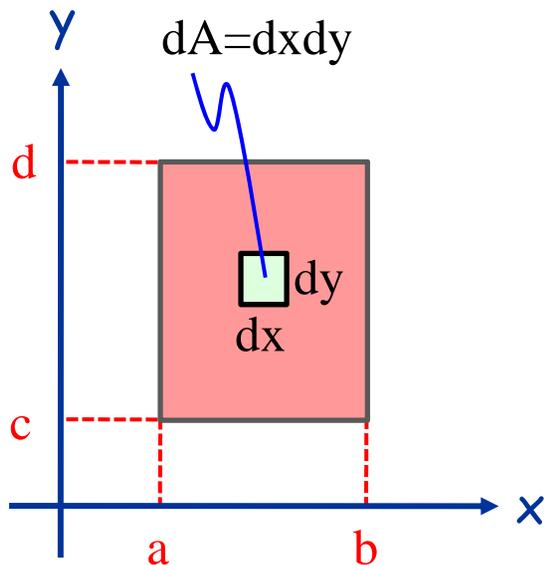


$$\begin{aligned} I_x &= \int y^2 dA = \int_c^d \int_a^b y^2 dx dy \\ &= \int_c^d \left[xy^2 \right]_a^b dy = \int_c^d (b-a)y^2 dy \\ &= \left[(b-a) \frac{y^3}{3} \right]_c^d \\ &= \frac{(b-a)(d^3 - c^3)}{3} \end{aligned}$$

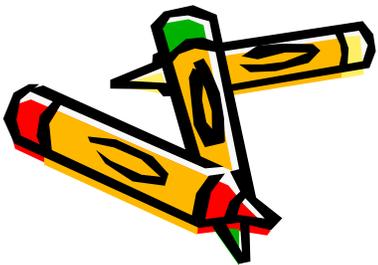


Determinação dos Momentos de Inércia por Integração Dupla

$$D = \{ (x, y) \mid a \leq x \leq b \text{ e } c \leq y \leq d \}$$

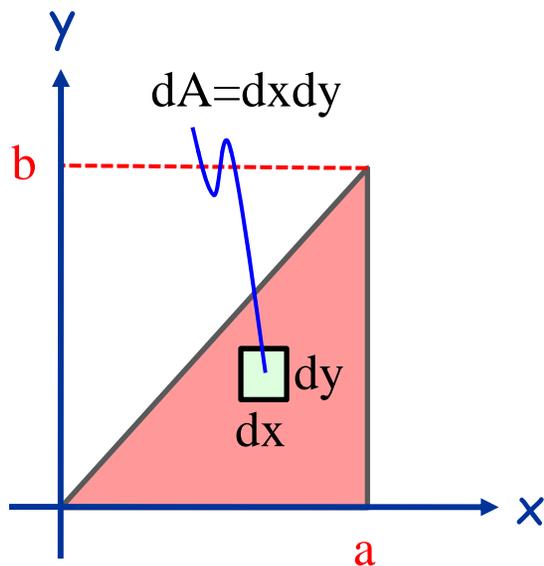


$$\begin{aligned} I_y &= \int x^2 dA = \int_c^d \int_a^b x^2 dx dy \\ &= \int_c^d \left[\frac{x^3}{3} \right]_a^b dy = \int_c^d \frac{b^3 - a^3}{3} dy \\ &= \left[\frac{b^3 - a^3}{3} y \right]_c^d \\ &= \frac{(b^3 - a^3)(d - c)}{3} \end{aligned}$$

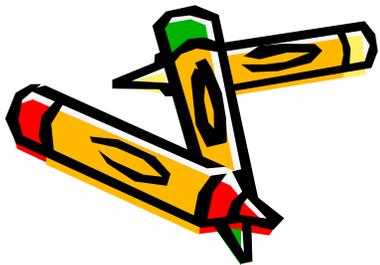


Determinação dos Momentos de Inércia por Integração Dupla

$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } 0 \leq y \leq \frac{b}{a}x \right\}$$

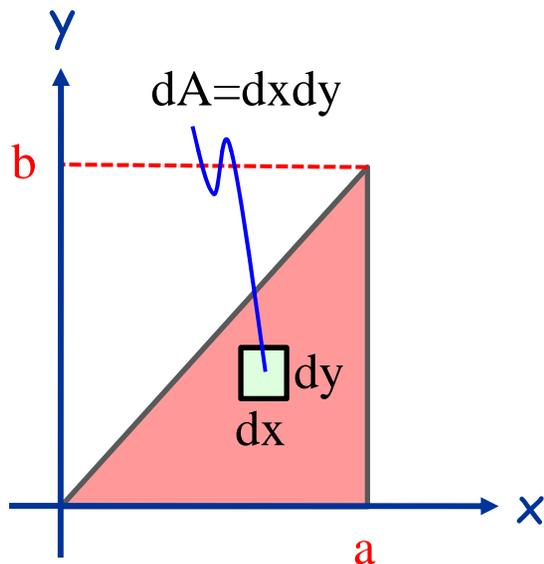


$$\begin{aligned} I_x &= \int y^2 dA = \int_0^a \int_0^{\frac{b}{a}x} y^2 dy dx \\ &= \int_0^a \left[\frac{y^3}{3} \right]_0^{\frac{b}{a}x} dx = \int_0^a \frac{b^3}{3a^3} x^3 dx \\ &= \left[\frac{b^3}{3a^3} \frac{x^4}{4} \right]_0^a = \frac{ab^3}{12} \end{aligned}$$



Determinação dos Momentos de Inércia por Integração Dupla

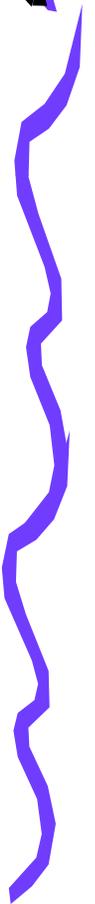
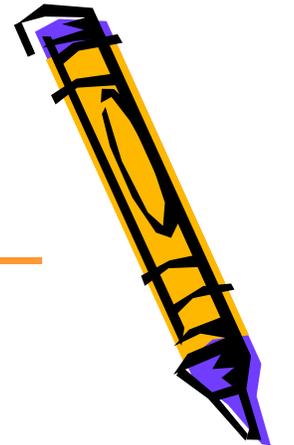
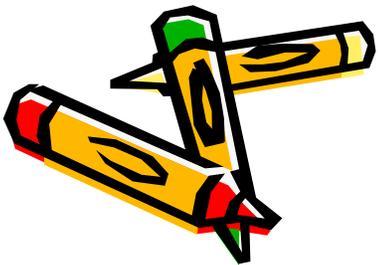
$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } 0 \leq y \leq \frac{b}{a}x \right\}$$



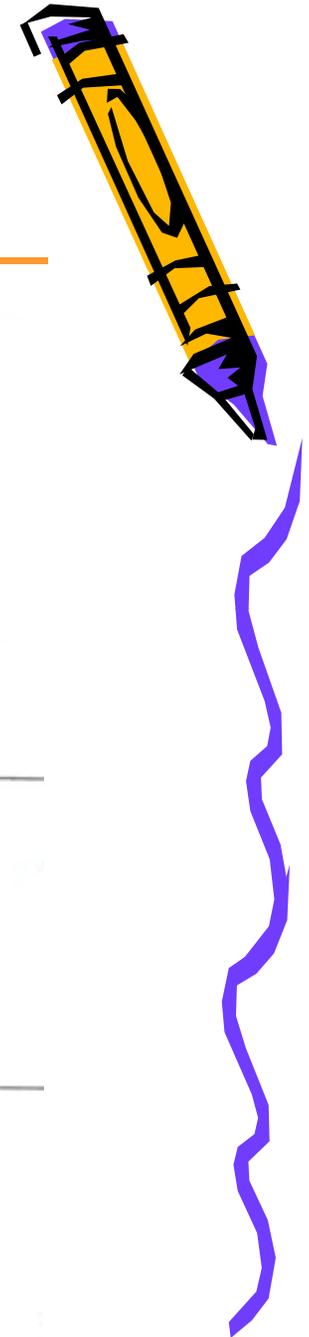
$$I_y = \int x^2 dA = \int_0^a \int_0^{\frac{b}{a}x} x^2 dy dx$$

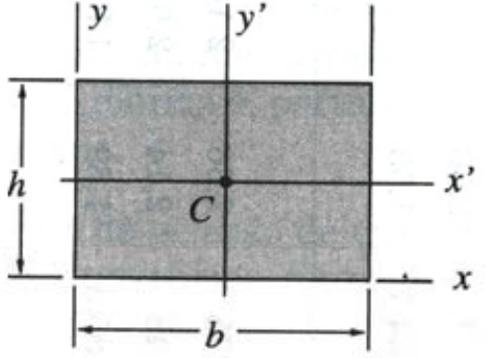
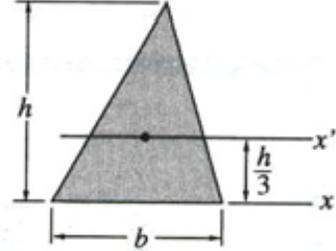
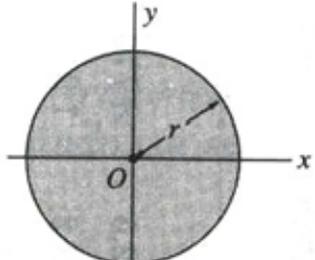
$$= \int_0^a \left[x^2 y \right]_0^{\frac{b}{a}x} dx = \int_0^a \frac{b}{a} x^3 dx$$

$$= \left[\frac{b}{a} \frac{x^4}{4} \right]_0^a = \frac{a^3 b}{4}$$

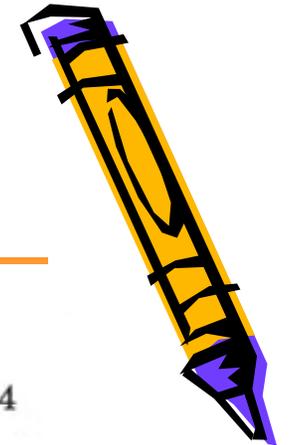


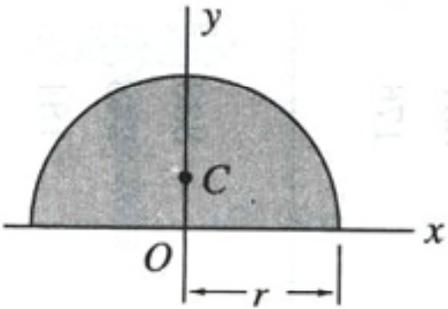
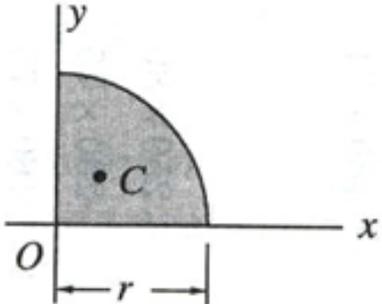
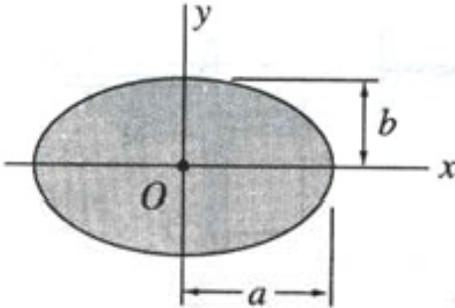
Momentos de Inércia de Figuras Geométricas Comuns



Retângulo		$\bar{I}_{x'} = \frac{1}{12} bh^3$ $\bar{I}_{y'} = \frac{1}{12} b^3 h$ $I_x = \frac{1}{3} bh^3$ $I_y = \frac{1}{3} b^3 h$
Triângulo		$\bar{I}_{x'} = \frac{1}{36} bh^3$ $I_x = \frac{1}{12} bh^3$
Círculo		$\bar{I}_x = \bar{I}_y = \frac{1}{4} \pi r^4$

Momentos de Inércia de Figuras Geométricas Comuns



Semicírculo		$I_x = \bar{I}_y = \frac{1}{8} \pi r^4$
Quadrante		$I_x = I_y = \frac{1}{16} \pi r^4$
Elipse		$\bar{I}_x = \frac{1}{4} \pi a b^3$ $\bar{I}_y = \frac{1}{4} \pi a^3 b$

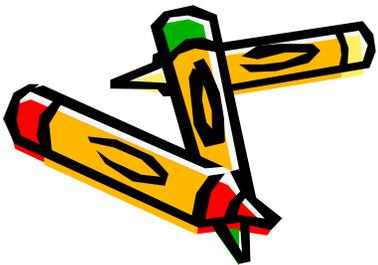
Determinação dos Momentos de Inércia por Integração de Fatias



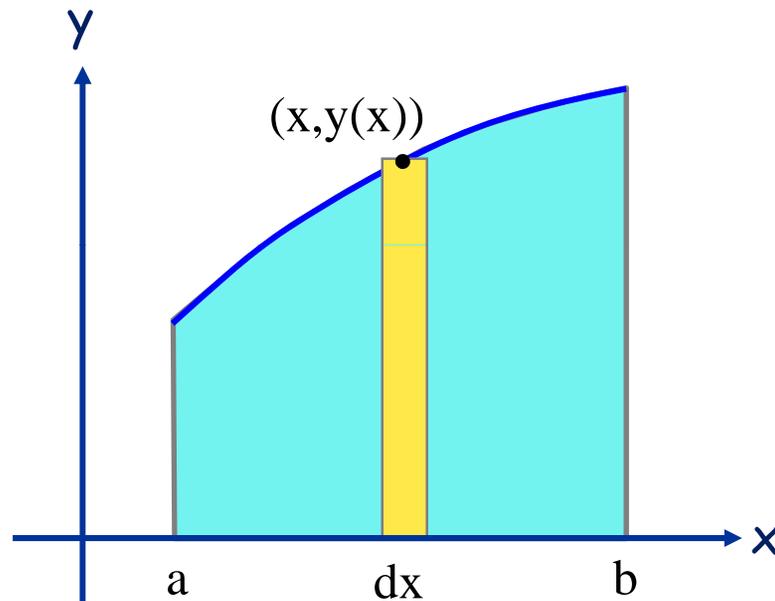
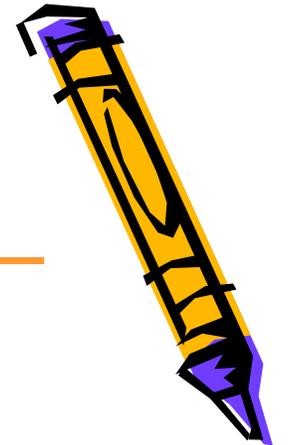
● $I_x = \int y^2 dA = \int dI_x^{el}$

$$I_y = \int x^2 dA = \int dI_y^{el}$$

- A idéia desta sistemática é considerar que a região de interesse é formada pela composição de infinitas fatias infinitesimais cujas formas correspondem a regiões cujo momento de inércia já é conhecido.



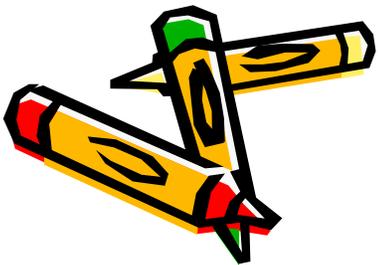
Determinação dos Momentos de Inércia por Integração de Fatias



$$\begin{aligned} I_x &= \int y^2 dA = \int dI_x^{\text{el}} \\ &= \int_a^b \frac{y(x)^3}{3} dx \end{aligned}$$

$$\begin{aligned} I_y &= \int x^2 dA = \int dI_y^{\text{el}} \\ &= \int_a^b x^2 y(x) dx \end{aligned}$$

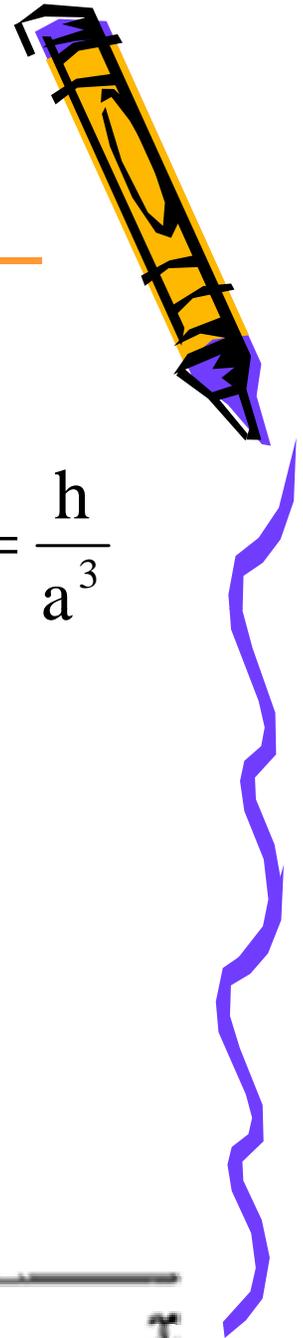
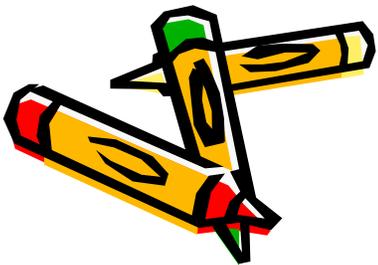
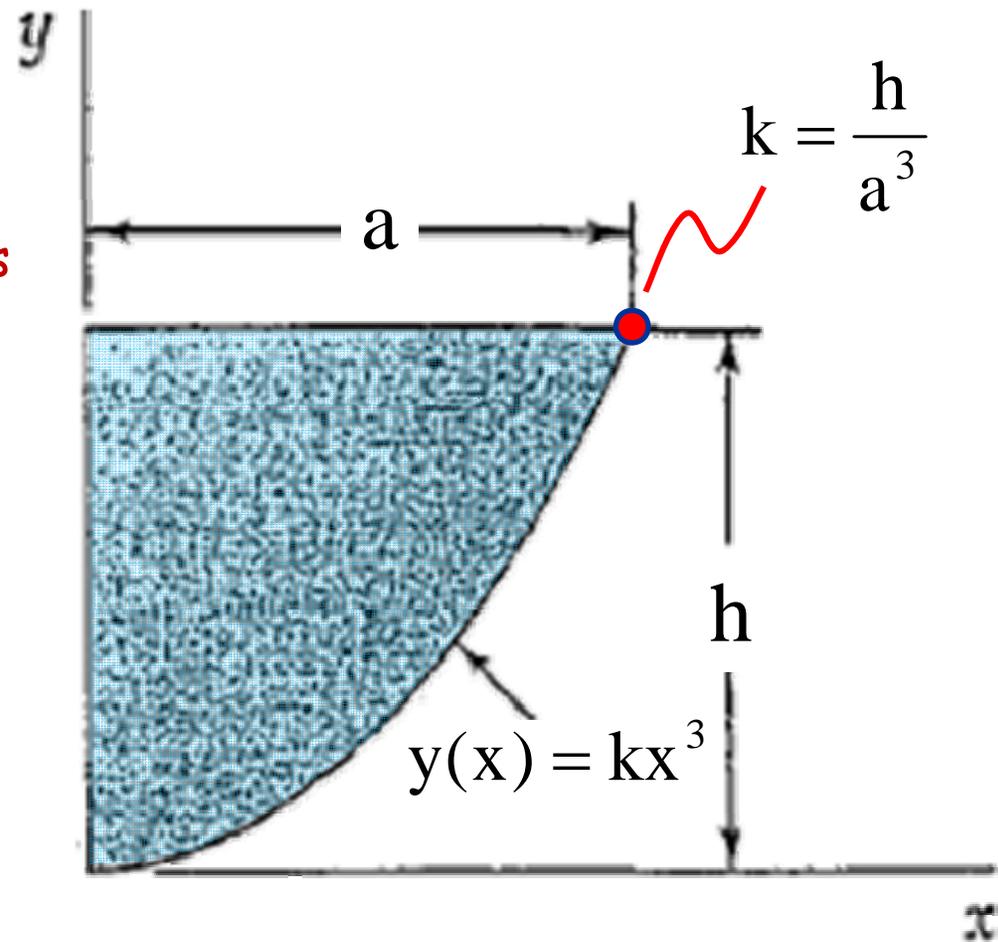
$$\begin{aligned} dI_x^{\text{el}} &= \frac{y(x)^3}{3} dx \\ dI_y^{\text{el}} &= x^2 y(x) dx \end{aligned}$$



Determinação dos Momentos de Inércia por Integração

Exemplo:

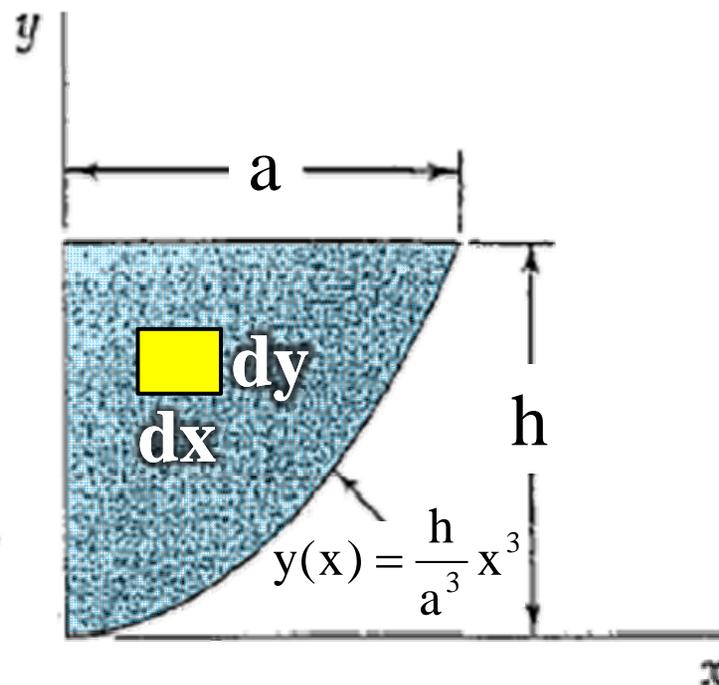
Determine por integração os momentos de inércia da superfície mostrada em relação aos eixos coordenados em termos de a e h .



Determinação dos Momentos de Inércia por Integração

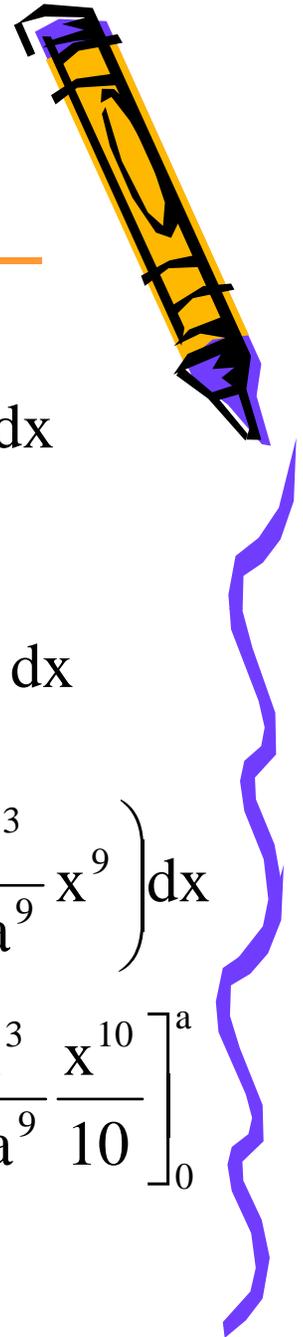
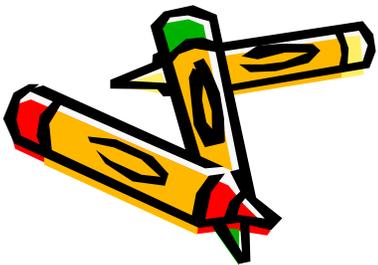
Exemplo (continuação):

Por integração dupla

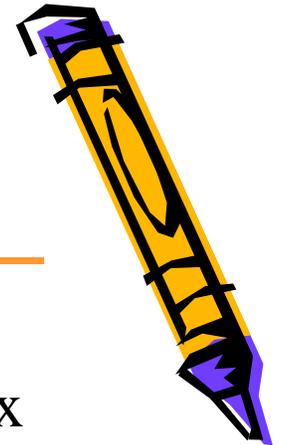


$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3} x^3 \leq y \leq h \right\}$$

$$\begin{aligned} I_x &= \int_0^a \int_{\frac{h}{a^3}x^3}^h y^2 dy dx \\ &= \int_0^a \left[\frac{y^3}{3} \right]_{\frac{h}{a^3}x^3}^h dx \\ &= \int_0^a \left(\frac{h^3}{3} - \frac{h^3}{3a^9} x^9 \right) dx \\ &= \left[\frac{h^3}{3} x - \frac{h^3}{3a^9} \frac{x^{10}}{10} \right]_0^a \\ &= \frac{3ah^3}{10} \end{aligned}$$

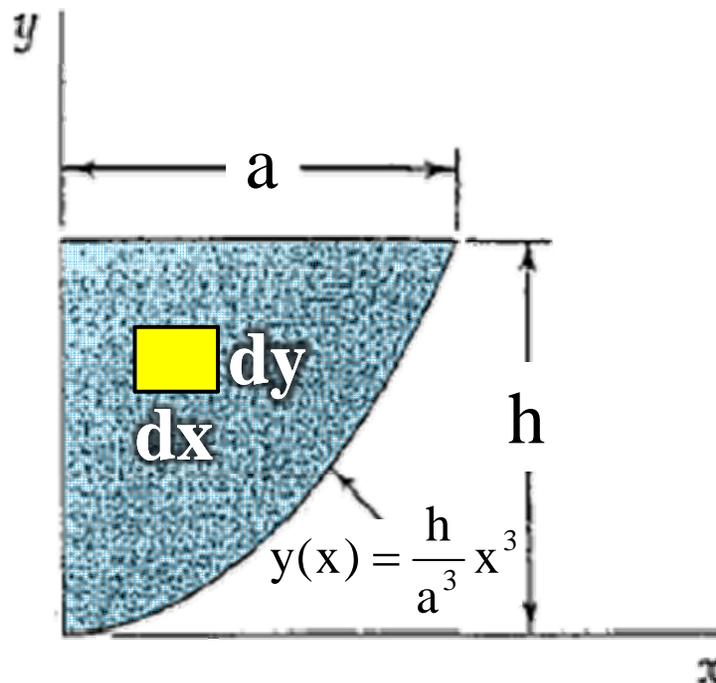


Determinação dos Momentos de Inércia por Integração



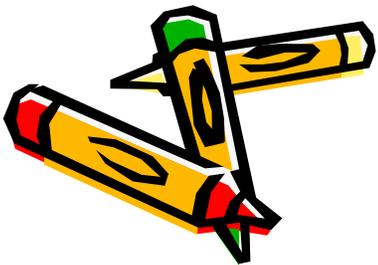
Exemplo (continuação):

Por integração dupla (cont.)



$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3} x^3 \leq y \leq h \right\}$$

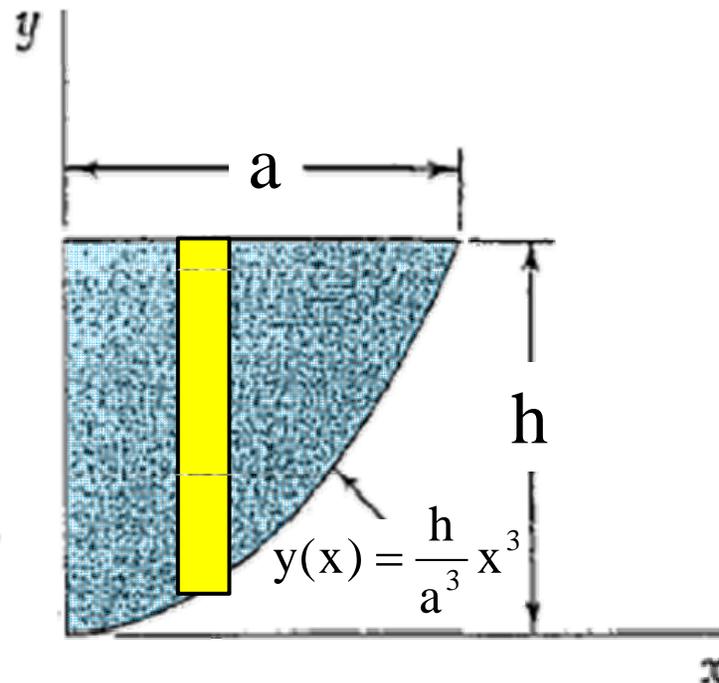
$$\begin{aligned} I_y &= \int_0^a \int_{\frac{h}{a^3}x^3}^h x^2 dy dx \\ &= \int_0^a \left[x^2 y \right]_{\frac{h}{a^3}x^3}^h dx \\ &= \int_0^a \left(hx^2 - \frac{h}{a^3} x^5 \right) dx \\ &= \left[\frac{hx^3}{3} - \frac{h}{a^3} \frac{x^6}{6} \right]_0^a \\ &= \frac{a^3 h}{6} \end{aligned}$$



Determinação dos Momentos de Inércia por Integração

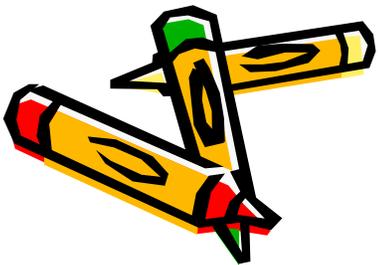
Exemplo (continuação):

Por integração de fatias



$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3} x^3 \leq y \leq h \right\}$$

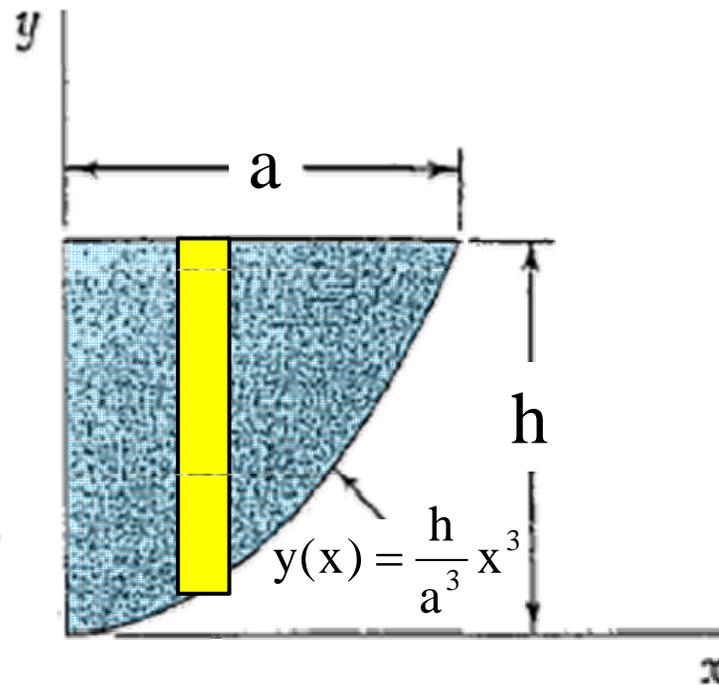
$$\begin{aligned} I_x &= \int dI_x \\ &= \int_0^a \left[\frac{h^3}{3} - \frac{y(x)^3}{3} \right] dx \\ &= \int_0^a \left[\frac{h^3}{3} - \frac{h^3 x^9}{3a^9} \right] dx \\ &= \left[\frac{h^3 x}{3} - \frac{h^3 x^{10}}{30a^9} \right]_0^a \\ &= \frac{3ah^3}{10} \end{aligned}$$



Determinação dos Momentos de Inércia por Integração

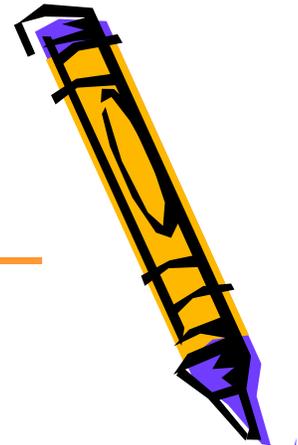
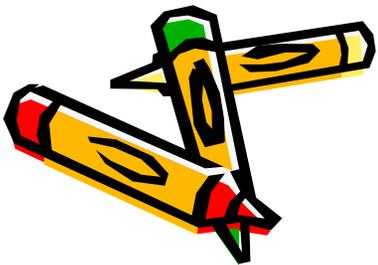
Exemplo (continuação):

Por integração de fatias



$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3}x^3 \leq y \leq h \right\}$$

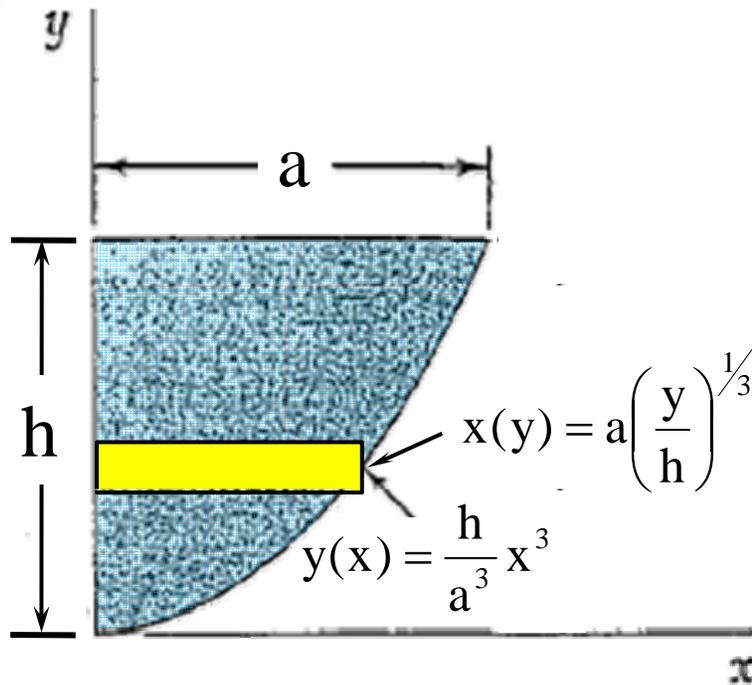
$$\begin{aligned} I_y &= \int dI_y \\ &= \int_0^a x^2 [h - y(x)] dx \\ &= \int_0^a x^2 \left(h - \frac{h}{a^3} x^3 \right) dx \\ &= \left[\frac{hx^3}{3} - \frac{hx^6}{6a^3} \right]_0^a \\ &= \frac{a^3 h}{6} \end{aligned}$$



Determinação dos Momentos de Inércia por Integração

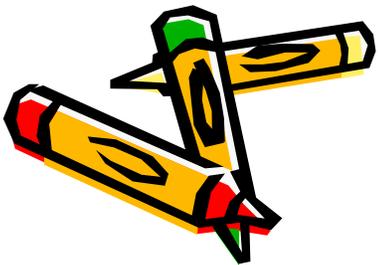
Exemplo (continuação):

Por integração de fatias



$$D = \left\{ (x, y) \mid 0 \leq y \leq h \text{ e } 0 \leq x \leq a \left(\frac{y}{h}\right)^{1/3} \right\}$$

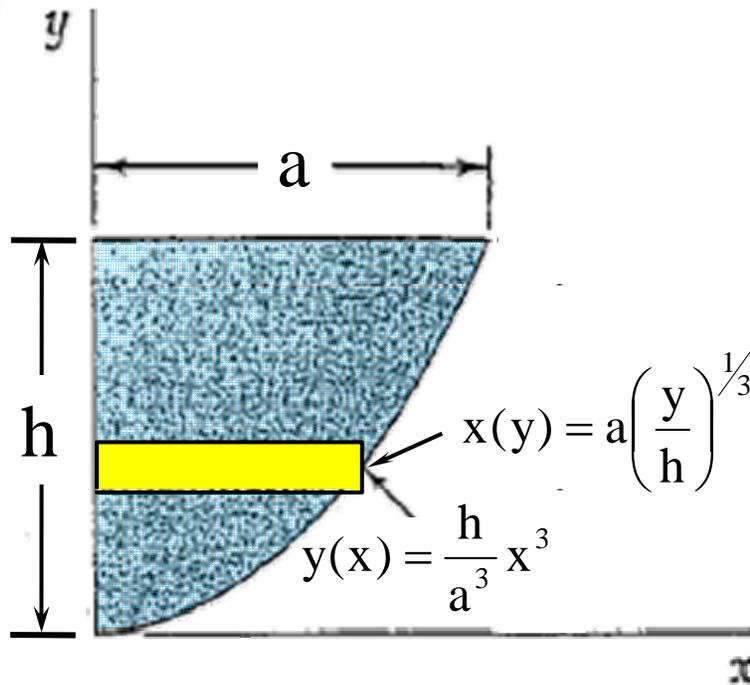
$$\begin{aligned} I_x &= \int dI_x \\ &= \int_0^h y^2 x(y) dy \\ &= \int_0^h a \frac{y^{7/3}}{h^{1/3}} dy \\ &= \left[a \frac{3}{10} \frac{y^{10/3}}{h^{1/3}} \right]_0^h \\ &= \frac{3}{10} ah^3 \end{aligned}$$



Determinação dos Momentos de Inércia por Integração

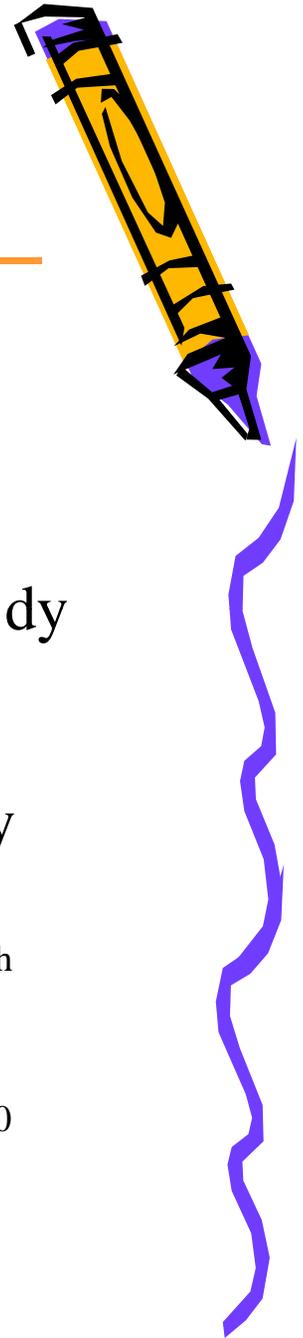
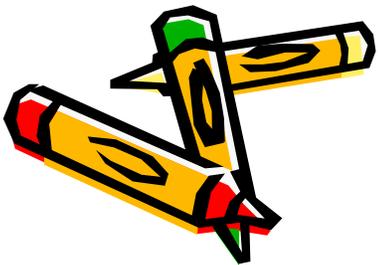
Exemplo (continuação):

Por integração de fatias

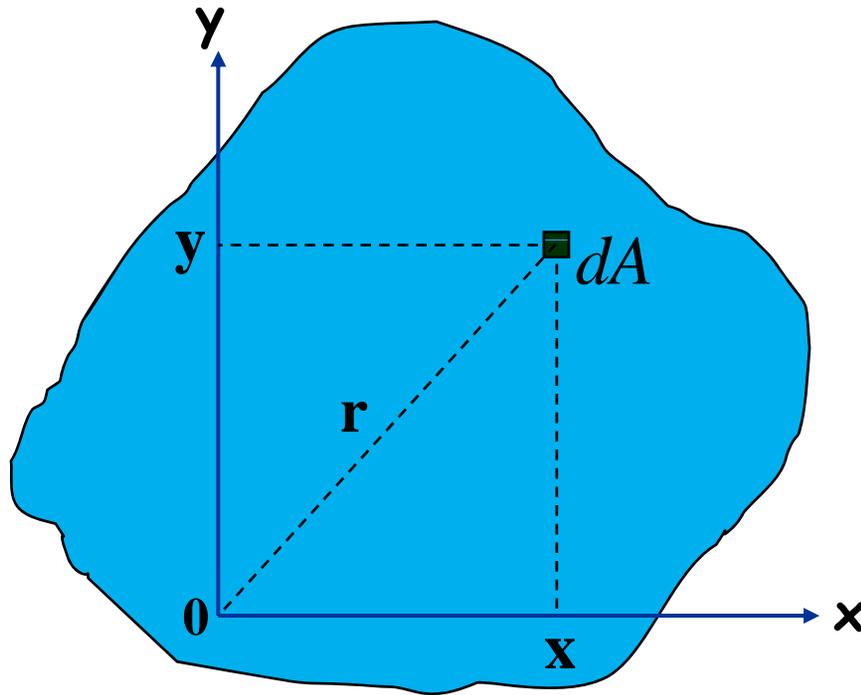


$$D = \left\{ (x, y) \mid 0 \leq y \leq h \text{ e } 0 \leq x \leq a\left(\frac{y}{h}\right)^{1/3} \right\}$$

$$\begin{aligned} I_y &= \int dI_y \\ &= \int_0^h \frac{x(y)^3}{3} dy \\ &= \int_0^h \frac{a^3 y}{3h} dy \\ &= \left[\frac{a^3 y^2}{6h} \right]_0^h \\ &= \frac{a^3 h}{6} \end{aligned}$$

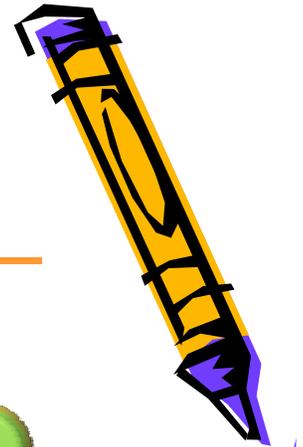
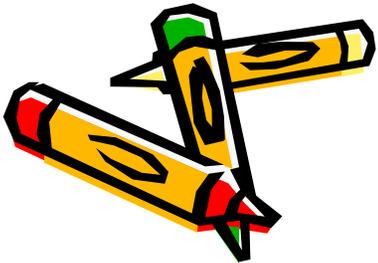
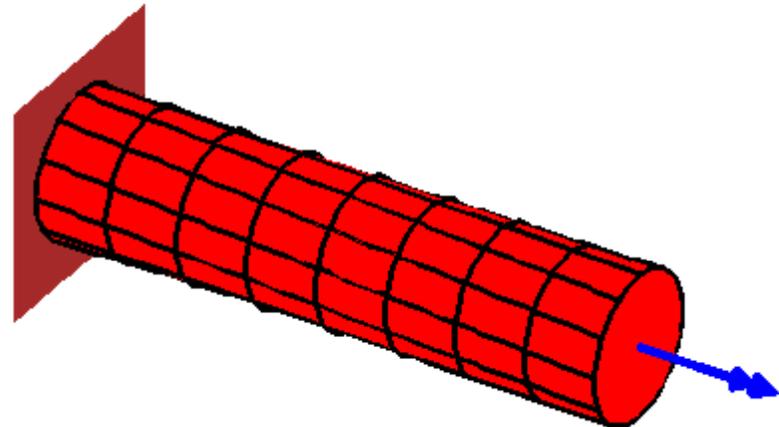


Momento Polar de Inércia

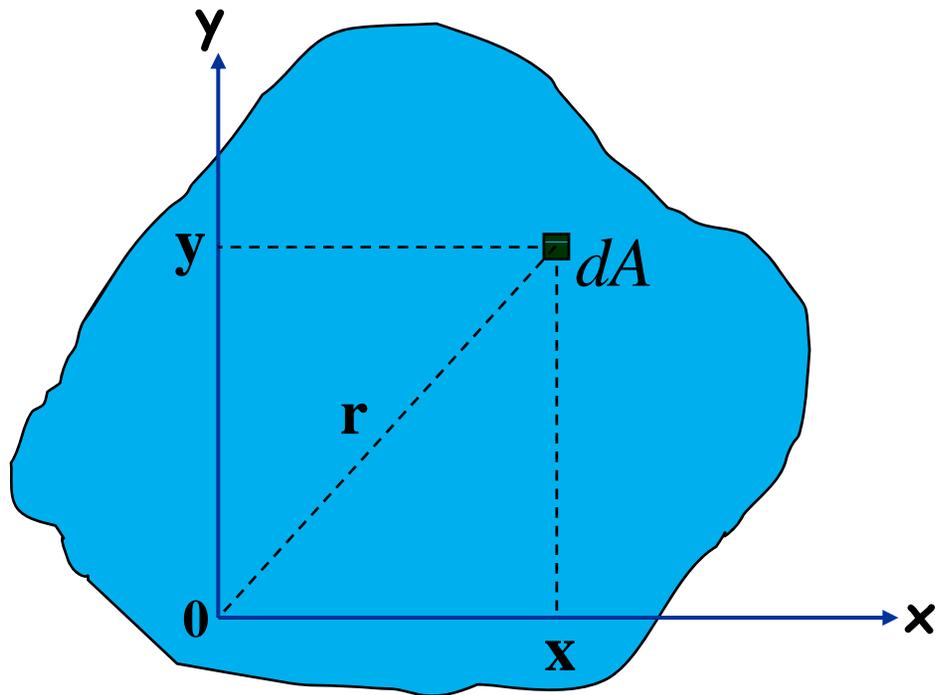
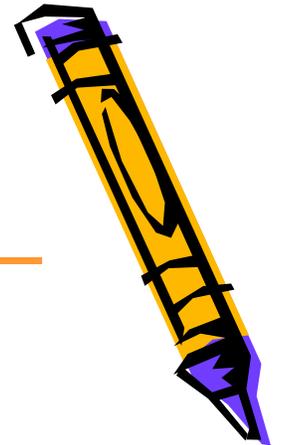


$$J_0 = \int_A r^2 dA$$

Presente no estudo de
barras sob torção



Momento Polar de Inércia e os Momentos de Inércia



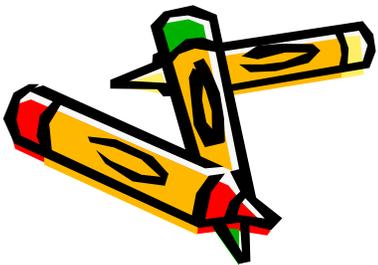
$$r^2 = y^2 + x^2$$



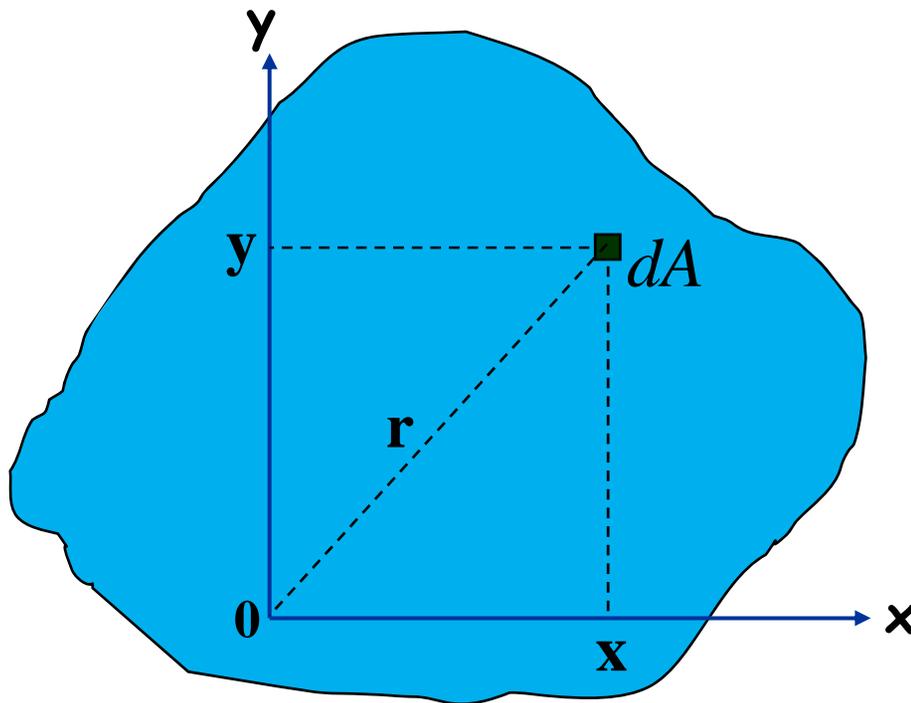
$$J_0 = \int_A (y^2 + x^2) dA$$

$$J_0 = \int_A y^2 dA + \int_A x^2 dA$$

$$J_0 = I_x + I_y$$



Raios de Giração

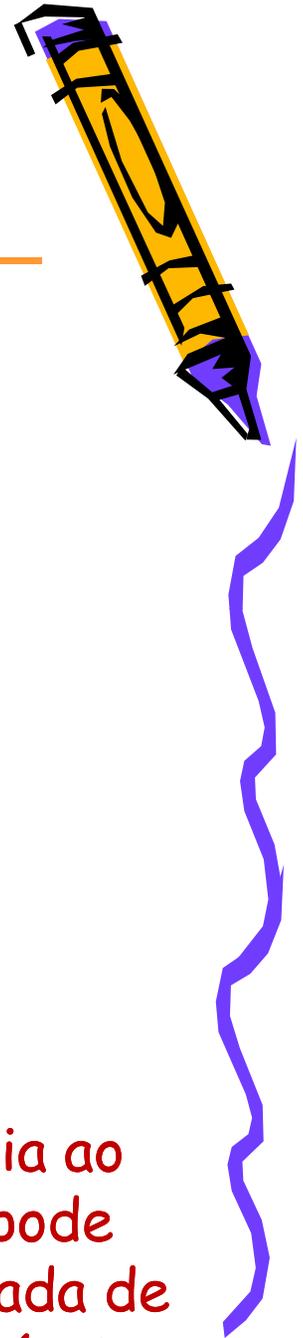
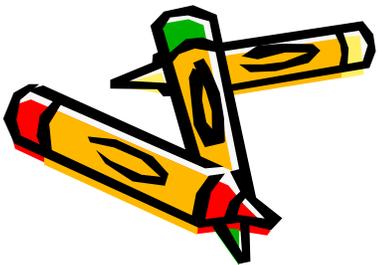


$$I_x = \int_A y^2 dA$$

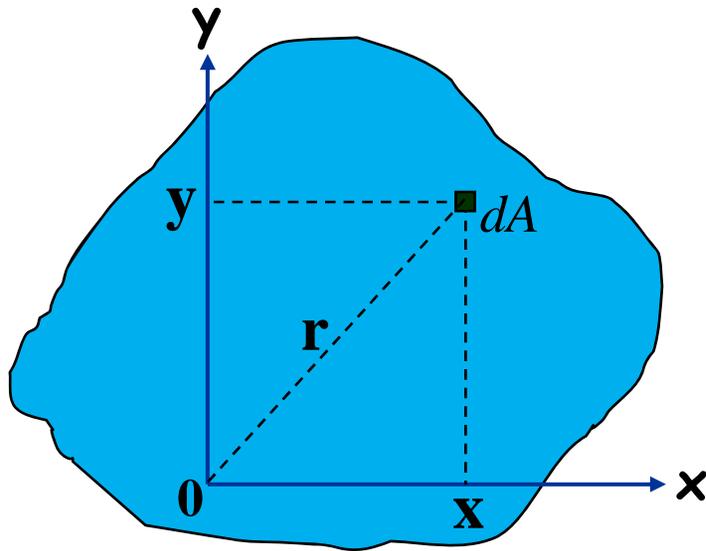
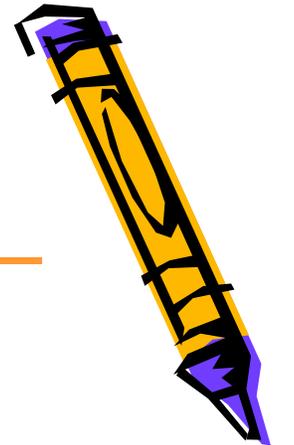
$$I_y = \int_A x^2 dA$$

$$J_0 = \int_A r^2 dA$$

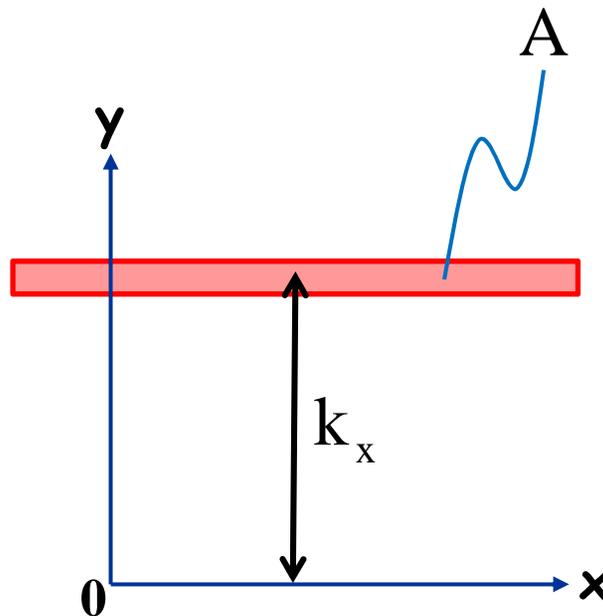
Cada raio de giração representa a distância ao eixo ou ponto correspondente na qual se pode concentrar toda a área da superfície estudada de modo que se tenha o mesmo momento de inércia.



Raios de Giração k_x



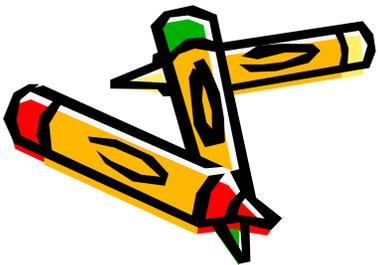
$$I_x = \int_A y^2 dA$$



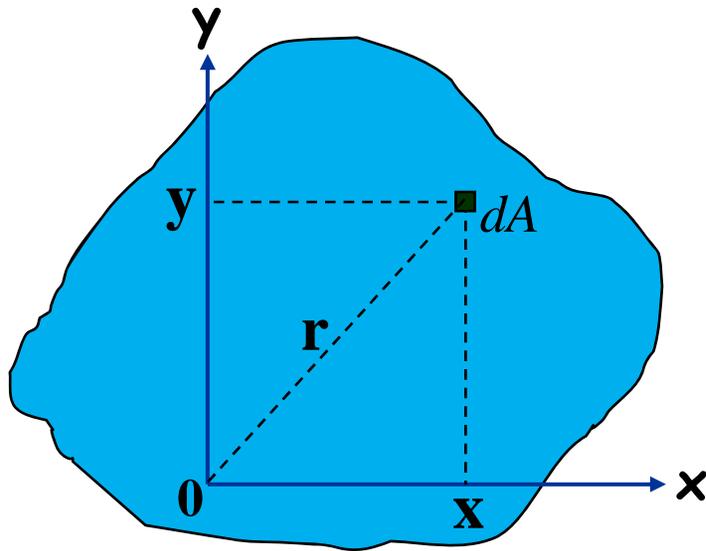
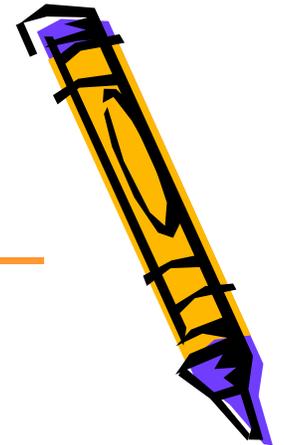
$$I_x = k_x^2 A$$



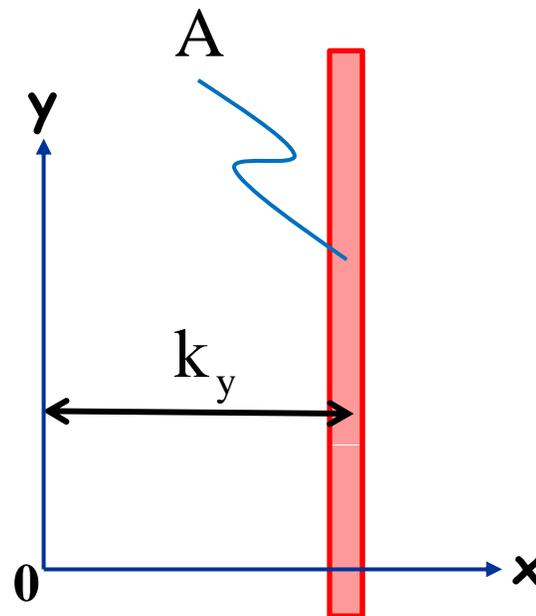
$$k_x = \sqrt{\frac{I_x}{A}}$$



Raios de Giração k_y



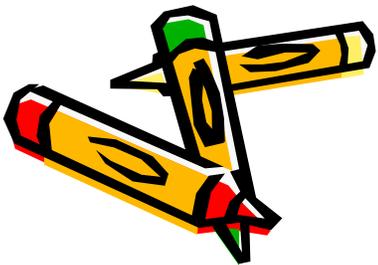
$$I_y = \int_A x^2 dA$$



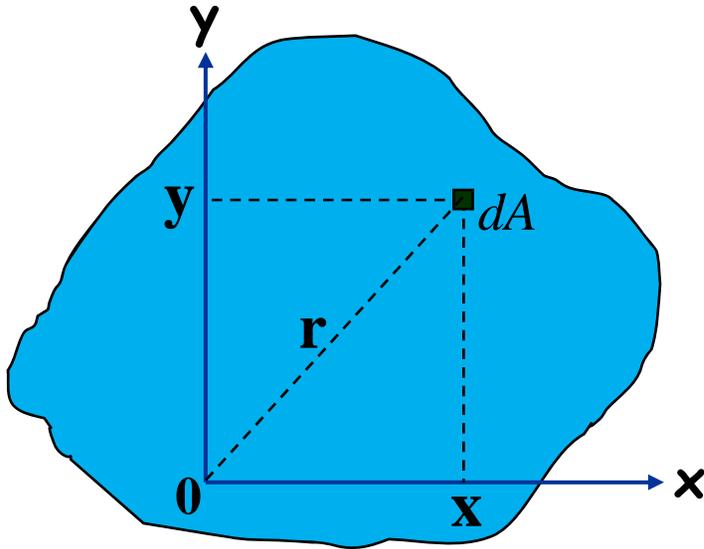
$$I_y = k_y^2 A$$



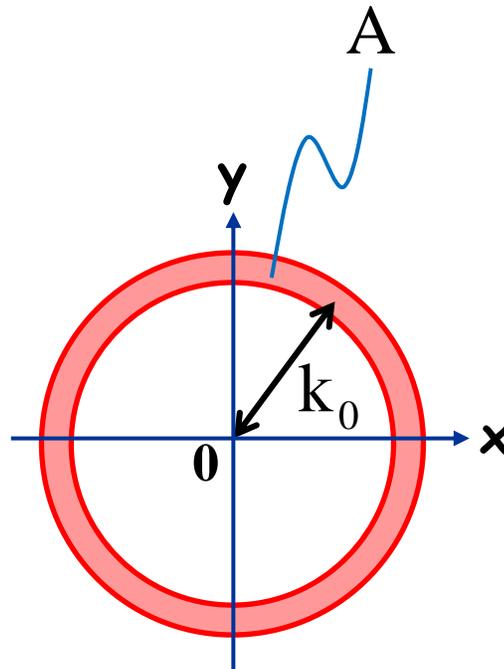
$$k_y = \sqrt{\frac{I_y}{A}}$$



Raios de Giração k_0



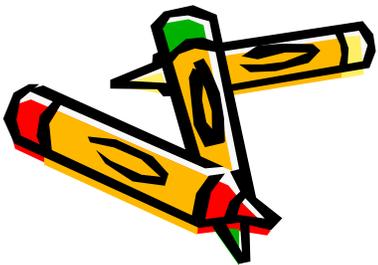
$$J_0 = \int_A r^2 dA$$



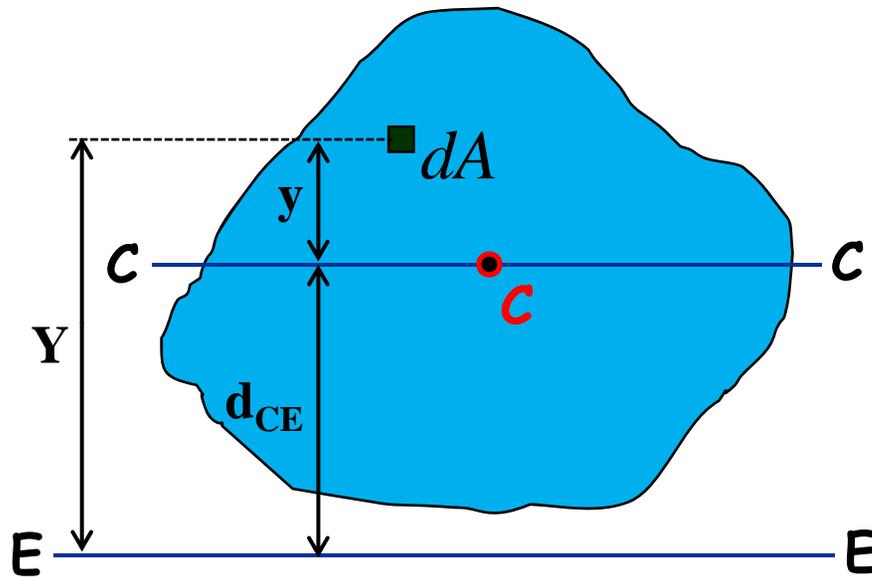
$$J_0 = k_0^2 A$$



$$k_0 = \sqrt{\frac{J_0}{A}}$$



Teorema dos Eixos Paralelos



$$I_{CC} = \int_A y^2 dA$$

$$I_{EE} = \int_A Y^2 dA$$

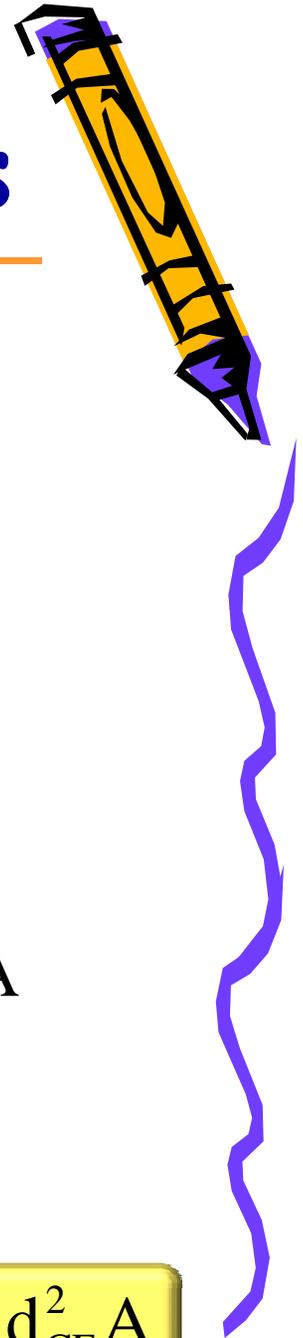
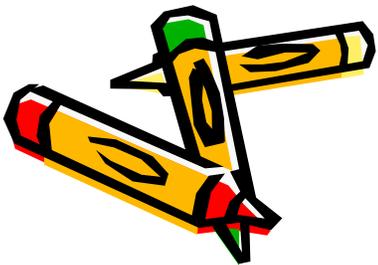
$$Y = y + d_{CE}$$

$$I_{EE} = \int_A (y + d_{CE})^2 dA = \int_A (y^2 + 2yd_{CE} + d_{CE}^2) dA$$

$$= \int_A y^2 dA + 2d_{CE} \int_A y dA + d_{CE}^2 \int_A dA$$

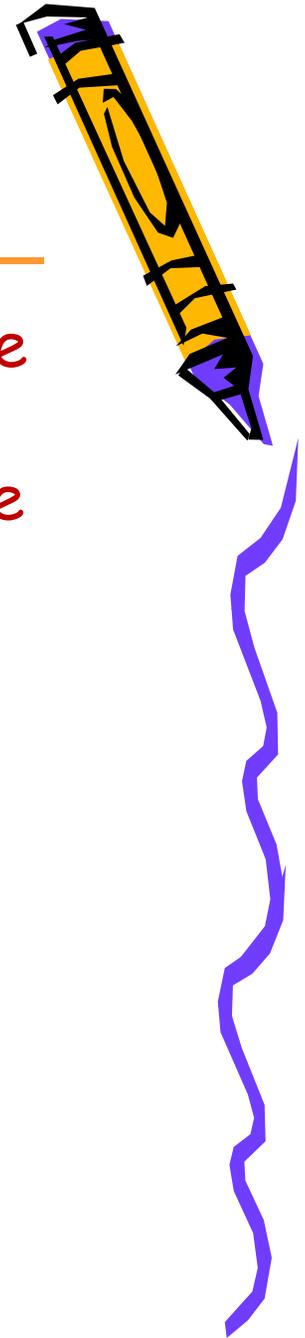
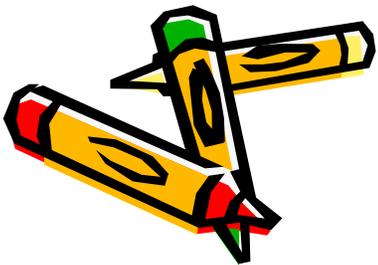
$$= I_{CC} + 2d_{CE} Q_{CC} + d_{CE}^2 A$$

$$I_{EE} = I_{CC} + d_{CE}^2 A$$



Momentos de Inércia de Superfícies Compostas

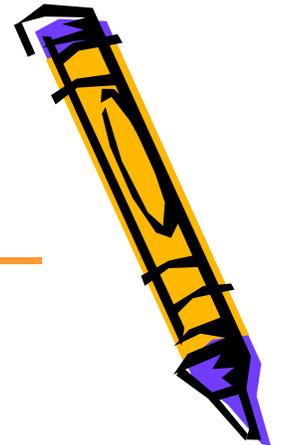
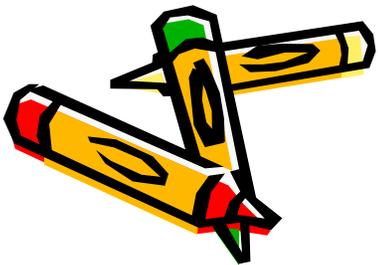
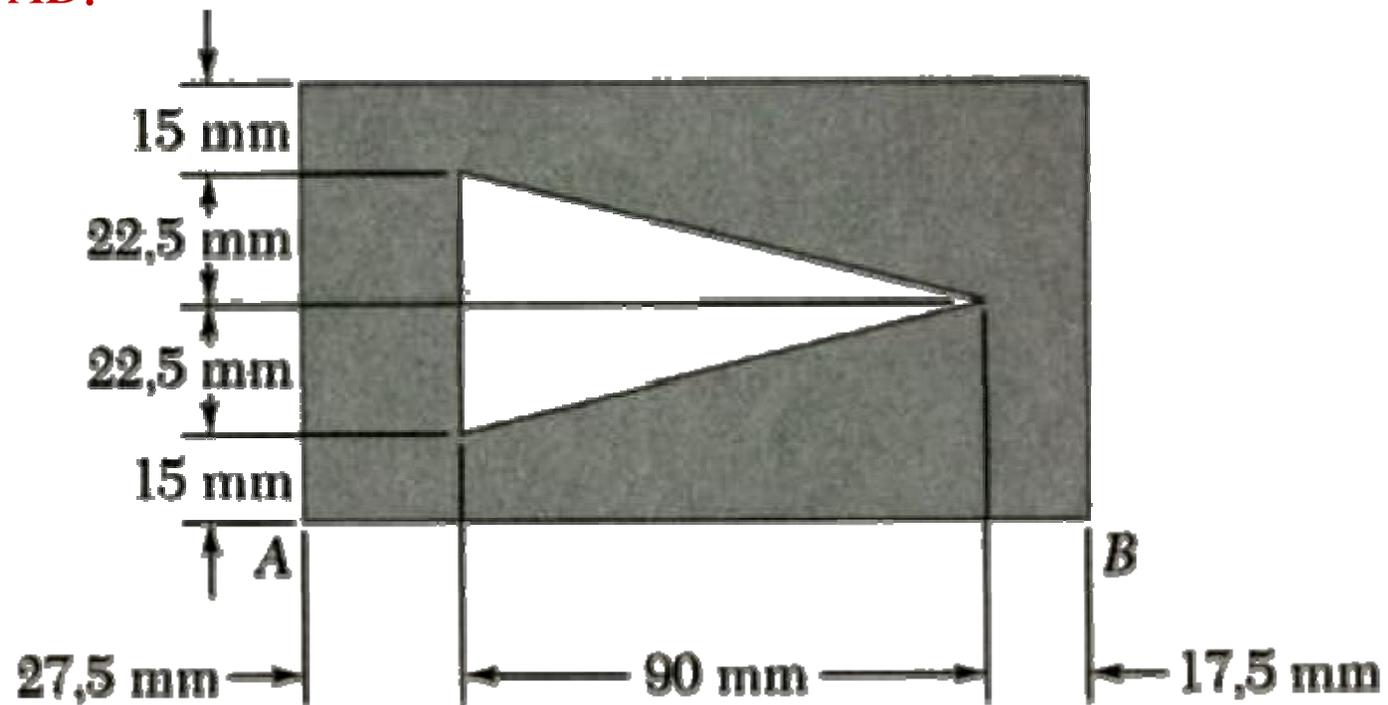
Quando se estiver interessado nos momentos de inércia de regiões que são identificadas como composições de regiões elementares, aplicam-se essas composições nas avaliações das integrais referentes às propriedades desejadas.



Momentos de Inércia

Exemplo:

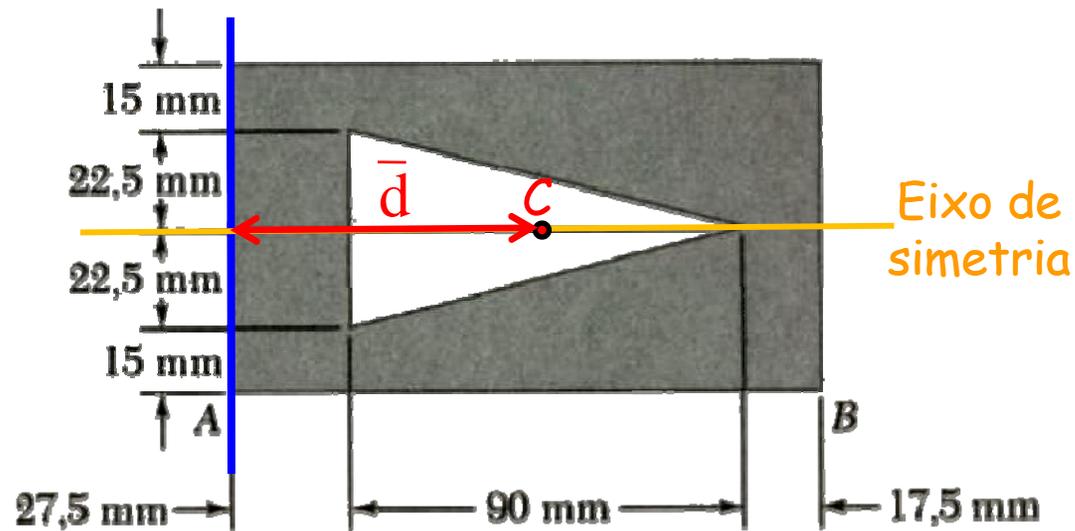
Determine os momentos de inércia da superfície mostrada em relação aos eixos centroidais paralelo e perpendicular ao lado AB .



Momentos de Inércia

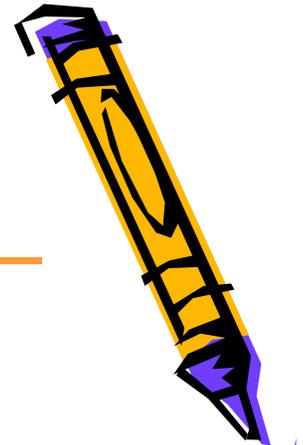
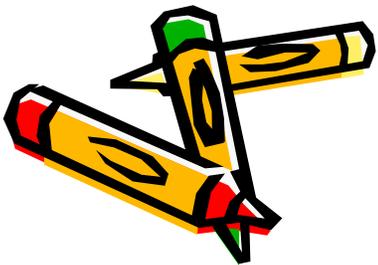
Exemplo (continuação):

Determinação do centróide



$$\bar{d} \left(135 \cdot 75 - \frac{45 \cdot 90}{2} \right) = \frac{135}{2} \cdot (135 \cdot 75) - \left(\frac{90}{3} + 27,5 \right) \cdot \frac{45 \cdot 90}{2}$$

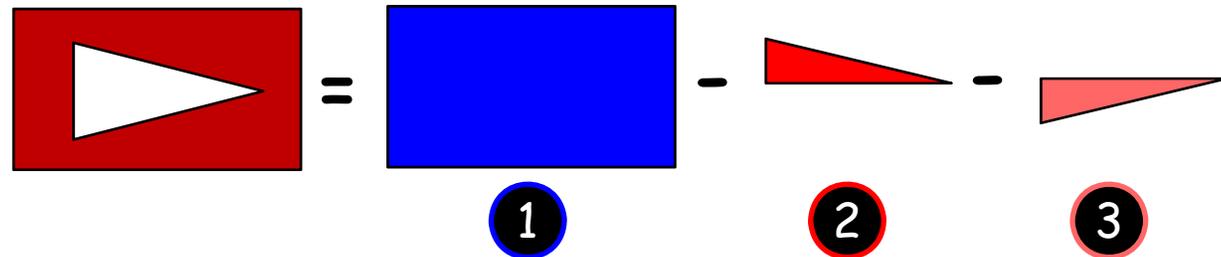
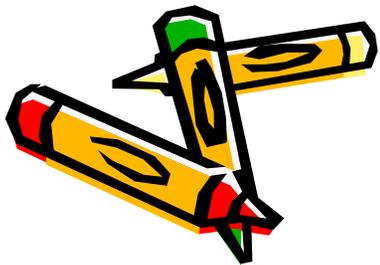
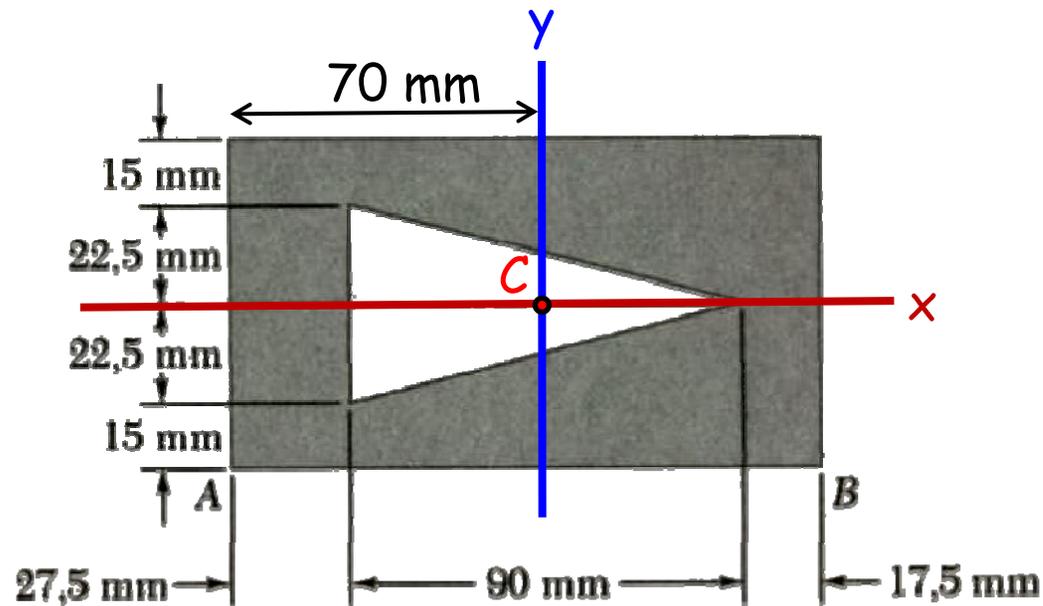
$$\bar{d} = 70 \text{ mm}$$



Momentos de Inércia

Exemplo (continuação):

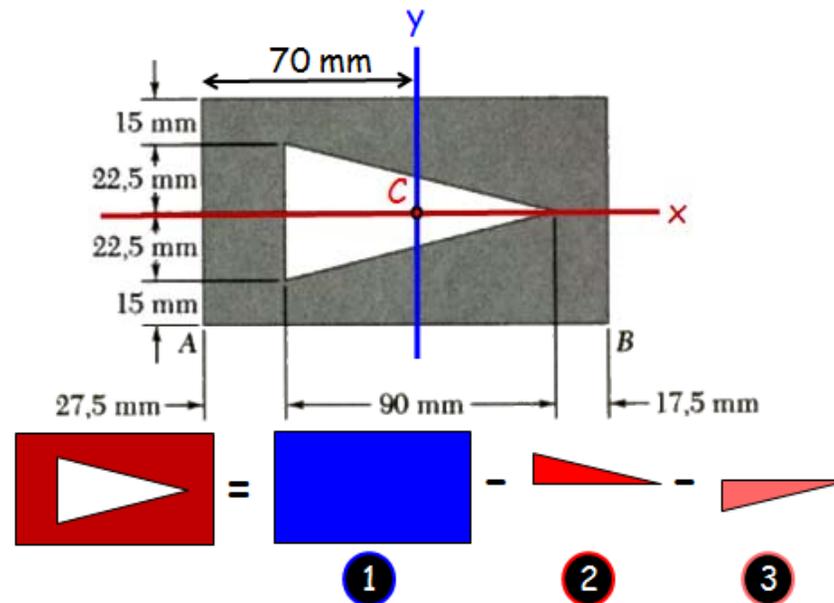
Determinação dos momentos de inércia



Momentos de Inércia

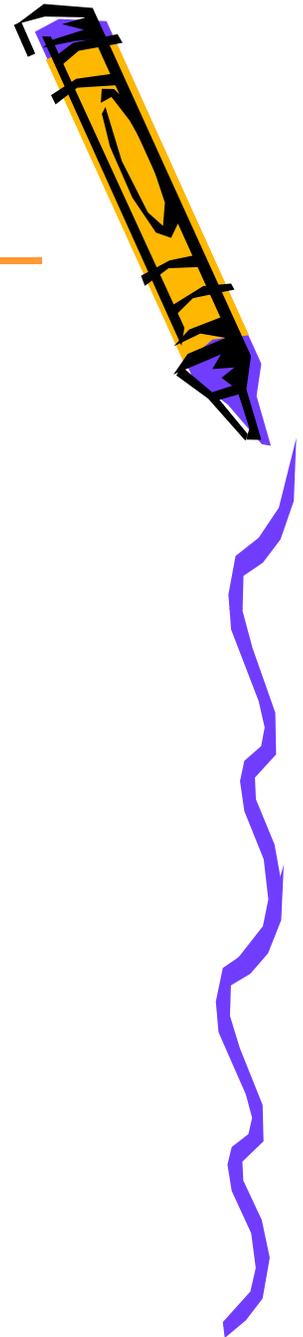
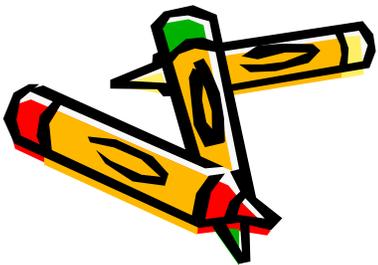
Exemplo (continuação):

Determinação dos momentos de inércia I_x



$$I_x = \frac{135 \cdot 75^3}{12} - \frac{90 \cdot 22,5^3}{12} - \frac{90 \cdot 22,5^3}{12}$$

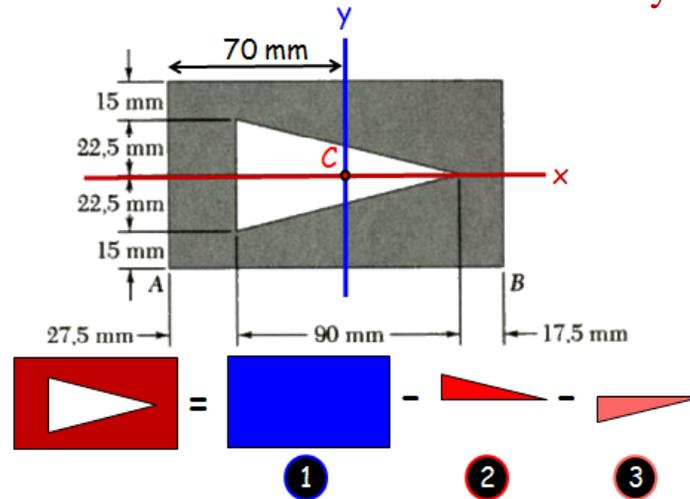
$$I_x = 4575234,4 \text{ mm}^4$$



Momentos de Inércia

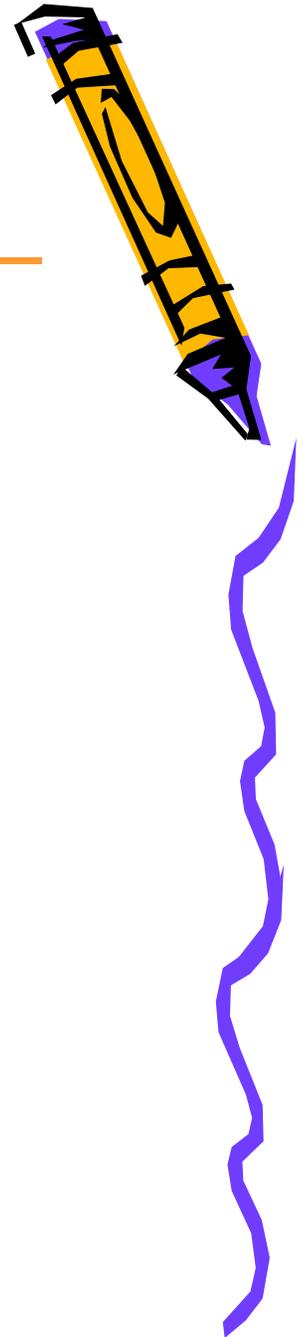
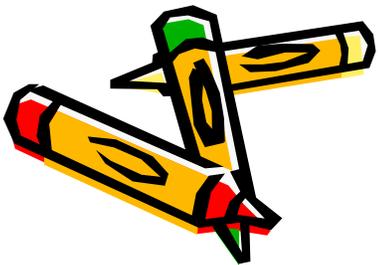
Exemplo (continuação):

Determinação dos momentos de inércia I_y

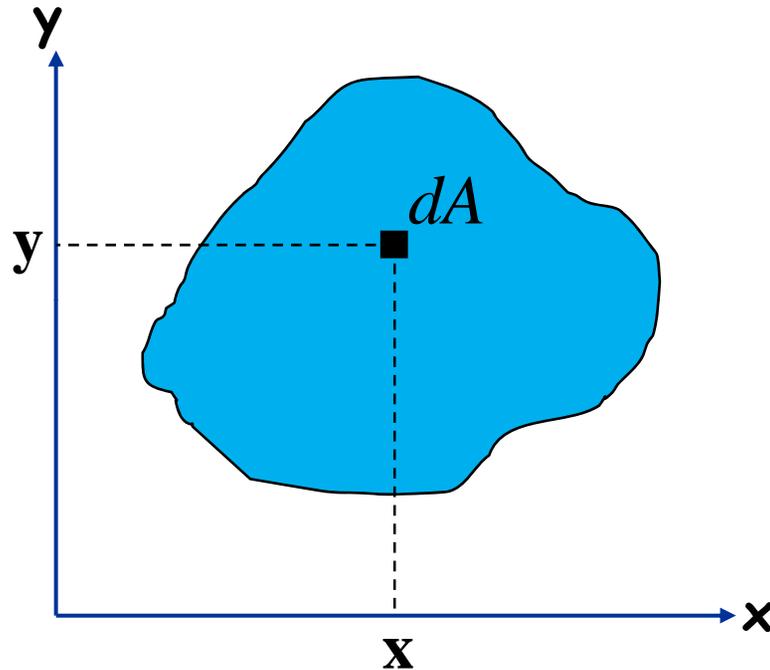


$$I_y = \left[\frac{75 \cdot 135^3}{12} + \left(70 - \frac{135}{2} \right)^2 (75 \cdot 135) \right] - 2 \left\{ \frac{22,5 \cdot 90^3}{36} + \left[70 - \left(\frac{90}{3} + 27,5 \right) \right]^2 \left[\frac{22,5 \cdot 90}{2} \right] \right\}$$

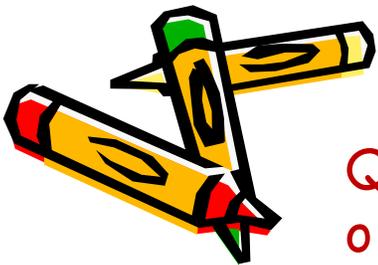
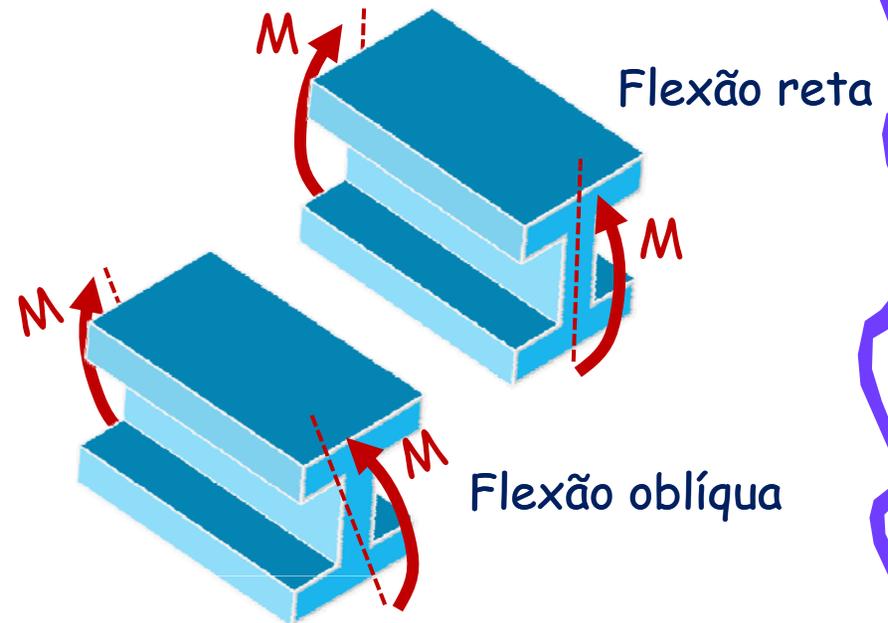
$$I_y = 14212968,8 \text{ mm}^4$$



Produto de Inércia



$$I_{xy} = \int_A xy dA$$

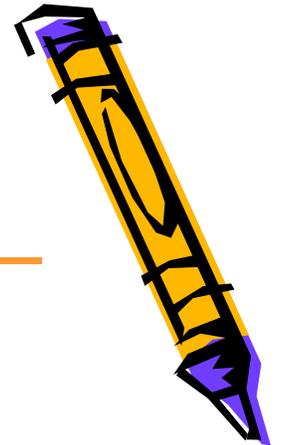
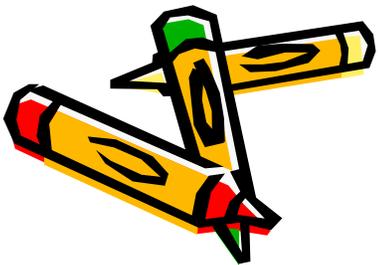


Quando pelo menos um dos eixos cartesianos é de simetria, o produto de inércia é nulo.

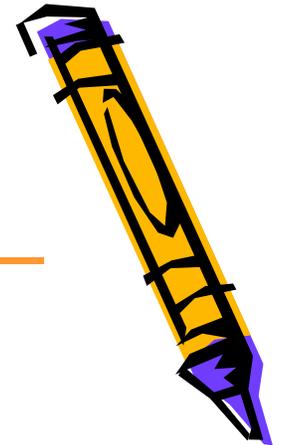
Determinação do Produto de Inércia por Integração

$$I_{xy} = \int xy dA$$

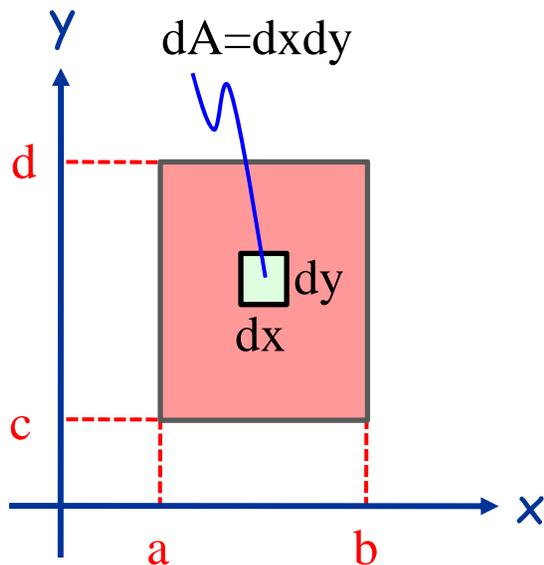
- Em princípio, para quantificação do produto de inércia, esse é calculado a partir de integral dupla no domínio representativo da região estudada, onde se deve escrever o elemento infinitesimal de área dA de acordo com a conveniência das coordenadas de descrição da região tratada.



Determinação do Produto de Inércia por Integração Dupla



$$D = \{ (x, y) \mid a \leq x \leq b \text{ e } c \leq y \leq d \}$$



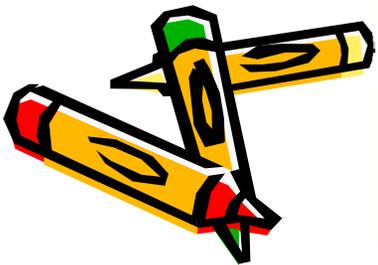
$$I_{xy} = \int \int xy dA = \int_c^d \int_a^b xy dx dy$$

$$= \int_c^d \left[\frac{x^2}{2} y \right]_a^b dy = \int_c^d \frac{b^2 - a^2}{2} y dy$$

$$= \left[\frac{(b^2 - a^2) y^2}{4} \right]_c^d$$

$$= \frac{(b^2 - a^2)(d^2 - c^2)}{4}$$

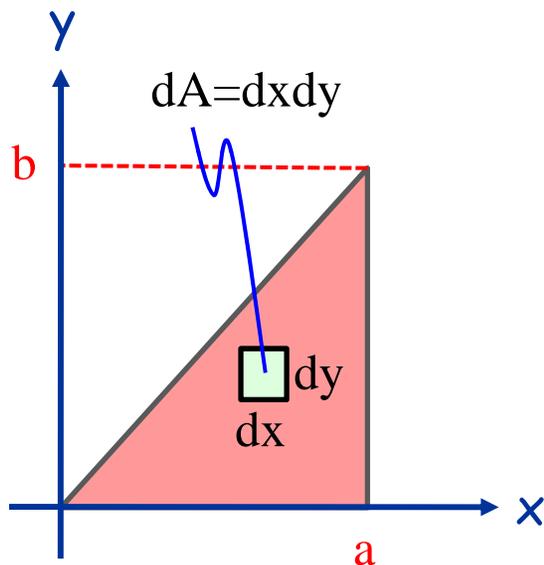
Neste caso, igual ao produto da área do retângulo pelas coordenadas do centróide do mesmo.



Determinação do Produto de Inércia por Integração Dupla

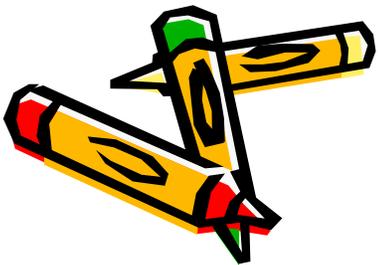


$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } 0 \leq y \leq \frac{b}{a}x \right\}$$

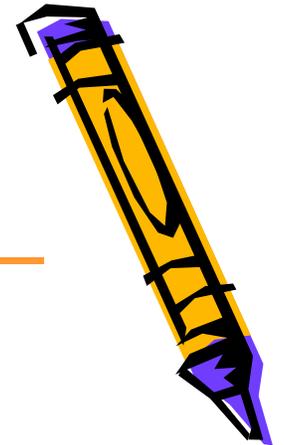


$$\begin{aligned} I_{xy} &= \int xy dA = \int_0^a \int_0^{\frac{b}{a}x} xy dy dx \\ &= \int_0^a \left[\frac{xy^2}{2} \right]_0^{\frac{b}{a}x} dx = \int_0^a \frac{b^2}{2a^2} x^3 dx \\ &= \left[\frac{b^2}{8a^2} x^4 \right]_0^a = \frac{a^2 b^2}{8} \end{aligned}$$

Em geral, não é igual ao produto da área pelas coordenadas do centróide da mesma.

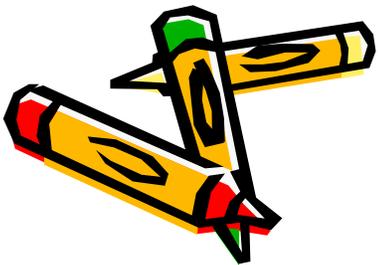


Determinação do Produto de Inércia por Integração de Fatias

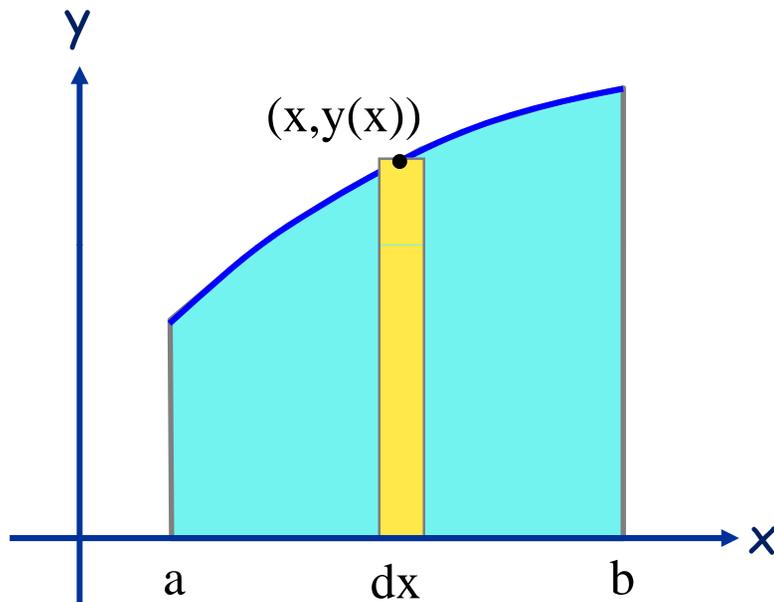
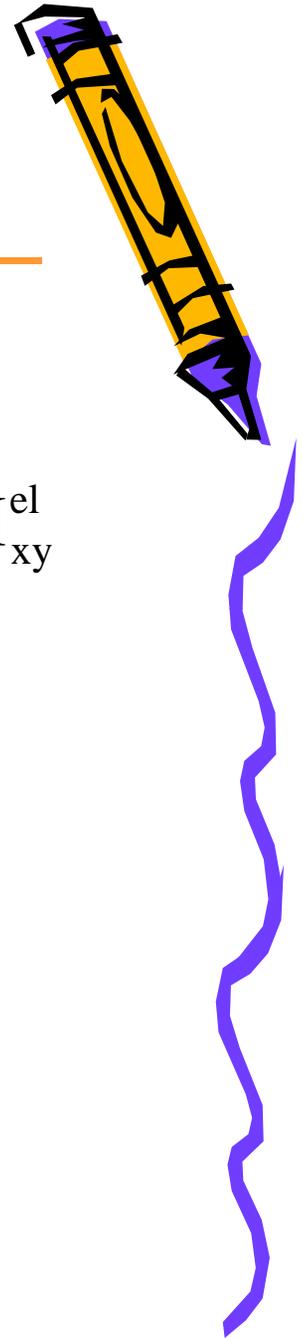


● $I_{xy} = \int xy dA = \int dI_{xy}^{el}$

- A idéia desta sistemática é considerar que a região de interesse é formada pela composição de infinitas fatias infinitesimais cujas formas correspondem a regiões cujo produto de inércia já é conhecido.

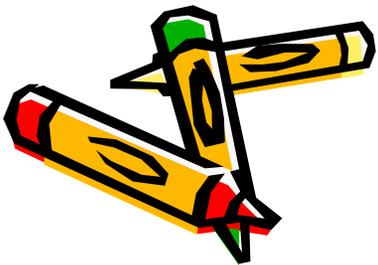


Determinação do Produto de Inércia por Integração de Fatias



$$I_{xy} = \int xy dA = \int dI_{xy}^{el}$$
$$= \int_a^b x \frac{y(x)^2}{2} dx$$

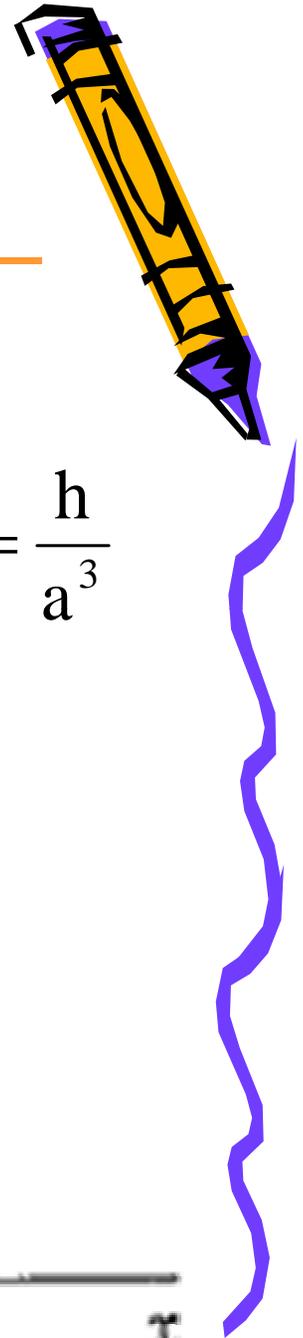
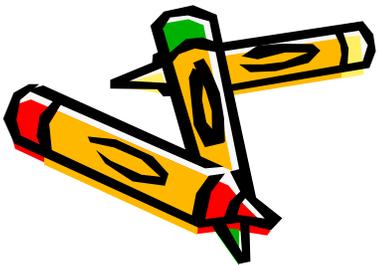
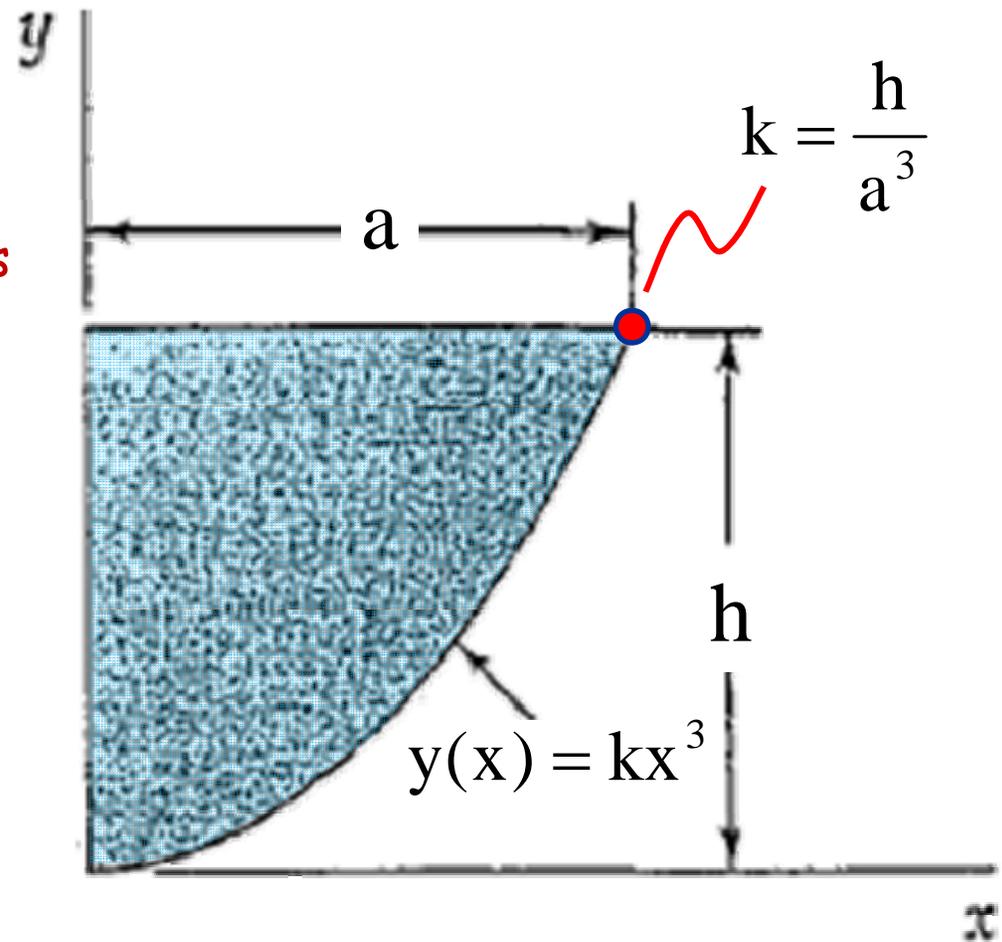
$$dI_{xy}^{el} = x \frac{y(x)^2}{2} dx$$



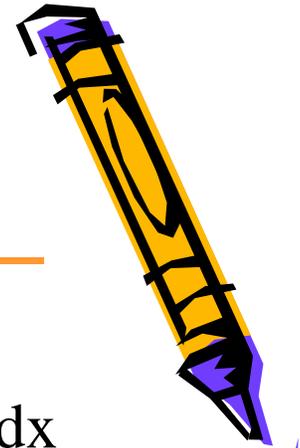
Determinação do Produto de Inércia por Integração

Exemplo:

Determine por integração o produto de inércia da superfície mostrada em relação aos eixos coordenados em termos de a e h .

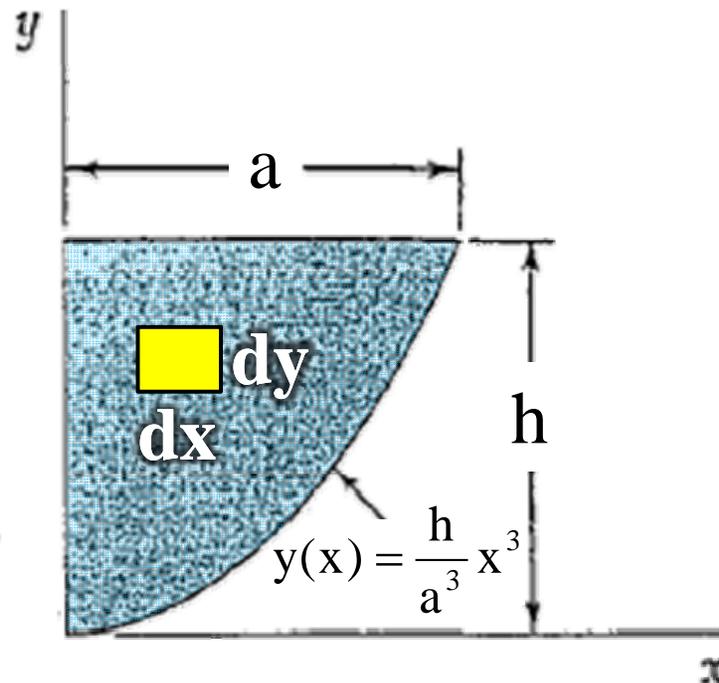


Determinação do Produto de Inércia por Integração



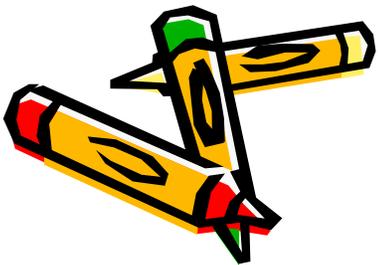
Exemplo (continuação):

Por integração dupla

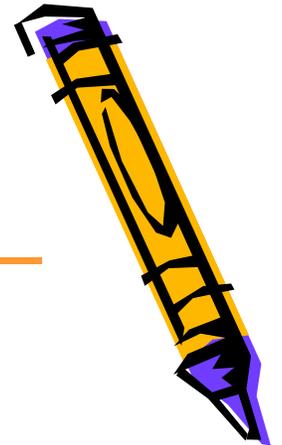


$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3} x^3 \leq y \leq h \right\}$$

$$\begin{aligned} I_{xy} &= \int_0^a \int_{\frac{h}{a^3}x^3}^h xy \, dy \, dx \\ &= \int_0^a \left[x \frac{y^2}{2} \right]_{\frac{h}{a^3}x^3}^h dx \\ &= \int_0^a \left(\frac{h^2 x}{2} - \frac{h^2 x^7}{2a^6} \right) dx \\ &= \left[\frac{h^2 x^2}{4} - \frac{h^2 x^8}{16a^6} \right]_0^a \\ &= \frac{3a^2 h^2}{16} \end{aligned}$$

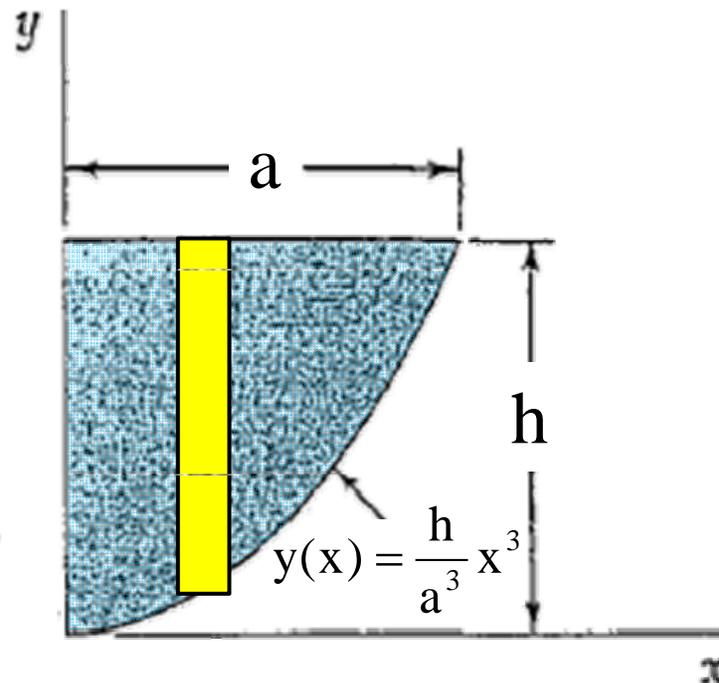


Determinação do Produto de Inércia por Integração



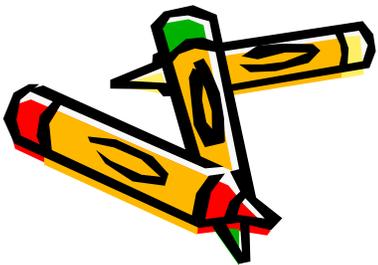
Exemplo (continuação):

Por integração de fatias



$$D = \left\{ (x, y) \mid 0 \leq x \leq a \text{ e } \frac{h}{a^3} x^3 \leq y \leq h \right\}$$

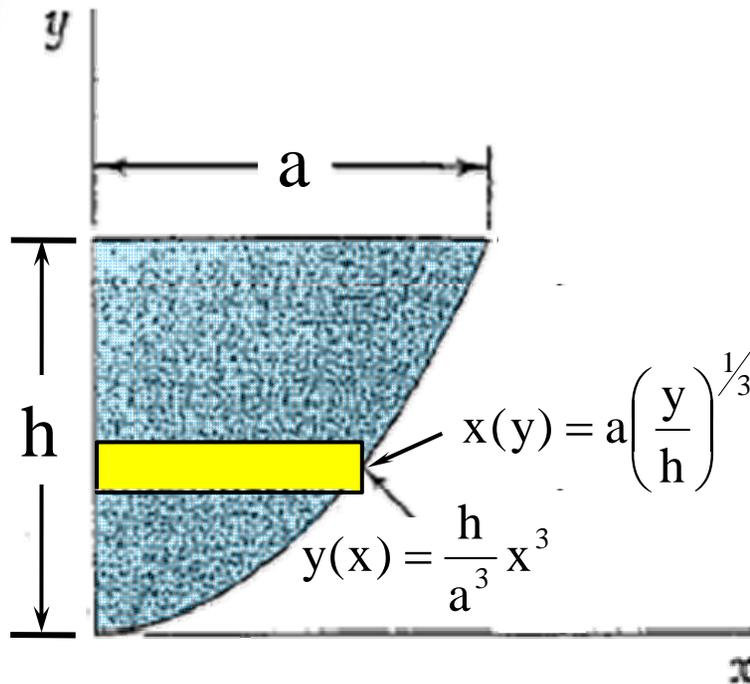
$$\begin{aligned} I_{xy} &= \int dI_{xy} \\ &= \int_0^a \left[h^2 - y(x)^2 \right] \frac{x}{2} dx \\ &= \int_0^a \left[\frac{h^2 x}{2} - \frac{h^2 x^7}{2a^6} \right] dx \\ &= \left[\frac{h^2 x^2}{4} - \frac{h^2 x^8}{16a^6} \right]_0^a \\ &= \frac{3a^2 h^2}{16} \end{aligned}$$



Determinação do Produto de Inércia por Integração

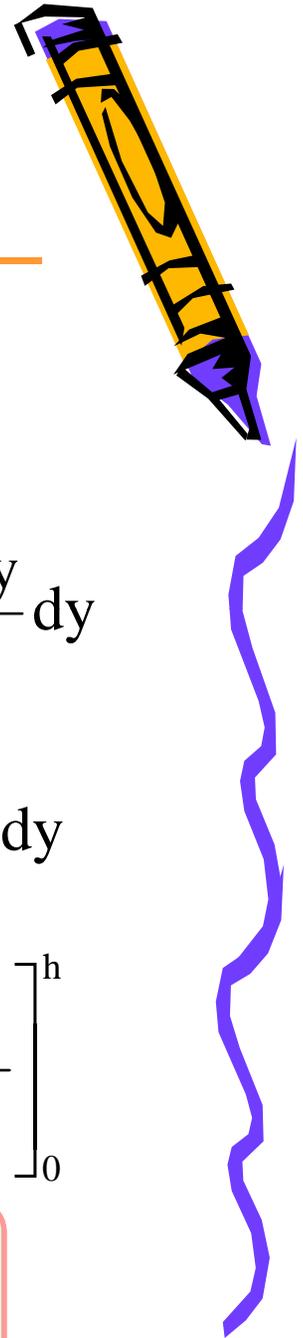
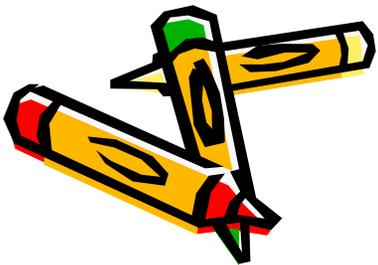
Exemplo (continuação):

Por integração de fatias

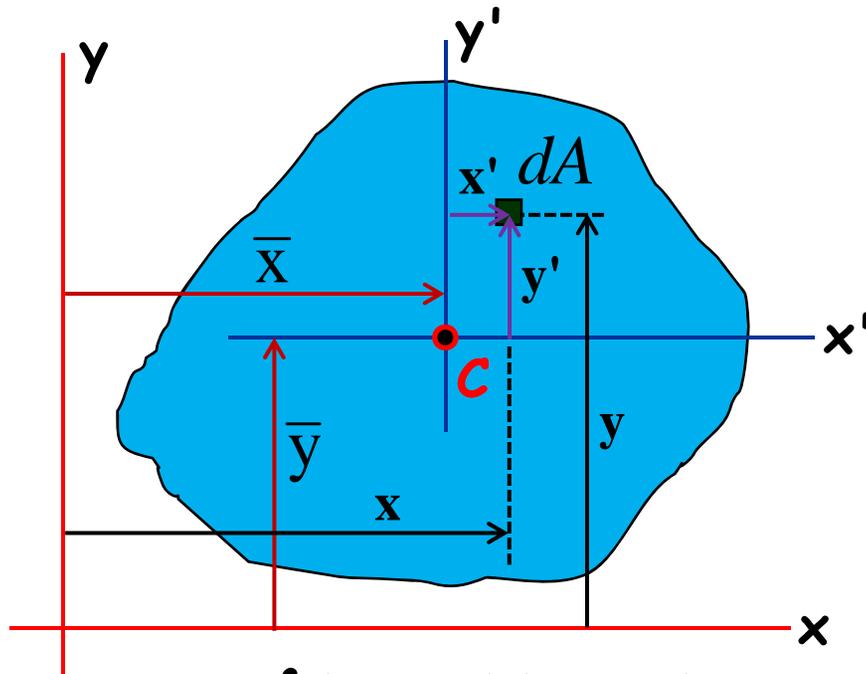


$$D = \left\{ (x, y) \mid 0 \leq y \leq h \text{ e } 0 \leq x \leq a \left(\frac{y}{h} \right)^{1/3} \right\}$$

$$\begin{aligned} I_{xy} &= \int dI_{xy} \\ &= \int_0^h \frac{x(y)^2 y}{2} dy \\ &= \int_0^h \frac{a^2 y^{5/3}}{2h^{2/3}} dy \\ &= \left[\frac{3a^2 y^{8/3}}{16h^{2/3}} \right]_0^h \\ &= \frac{3a^2 h^2}{16} \end{aligned}$$



Teorema dos Eixos Paralelos para o Produto de Inércia



$$I_{x'y'} = \int_A x'y'dA$$

$$I_{xy} = \int_A xydA$$

$$x = x' + \bar{x}$$

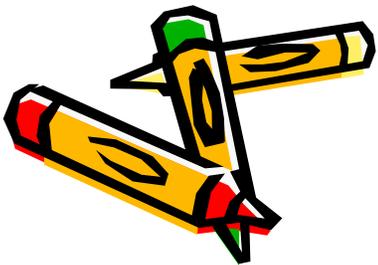
$$y = y' + \bar{y}$$

$$I_{xy} = \int_A (x' + \bar{x})(y' + \bar{y})dA = \int_A (x'y' + x'\bar{y} + \bar{x}y' + \bar{x}\bar{y})dA$$

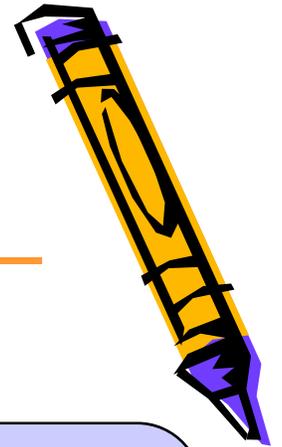
$$= \int_A x'y'dA + \bar{y} \int_A x'dA + \bar{x} \int_A y'dA + \bar{x}\bar{y} \int_A dA$$

$$= I_{x'y'} + \bar{y} \overset{0}{Q_{y'}} + \bar{x} \overset{0}{Q_{x'}} + \bar{x}\bar{y}A \rightarrow$$

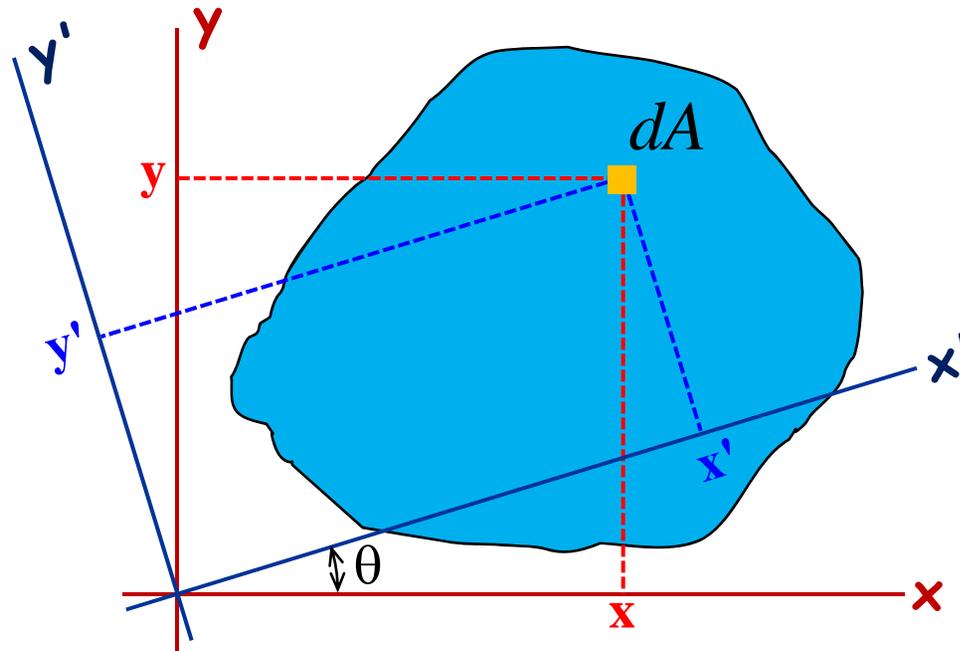
$$I_{xy} = I_{x'y'} + \bar{x}\bar{y}A$$



Eixos e Momentos de Inércia Principais



$$I_x = \int_A y^2 dA$$
$$I_y = \int_A x^2 dA$$
$$I_{xy} = \int_A xy dA$$



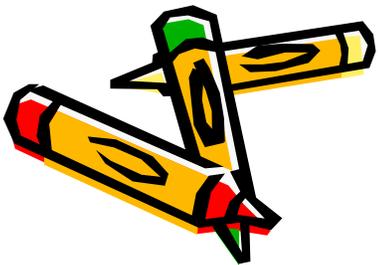
$$I_{x'} = \int_A y'^2 dA$$
$$I_{y'} = \int_A x'^2 dA$$
$$I_{x'y'} = \int_A x'y' dA$$

$$x' = x \cos \theta + y \sin \theta \quad \text{e} \quad y' = -x \sin \theta + y \cos \theta$$

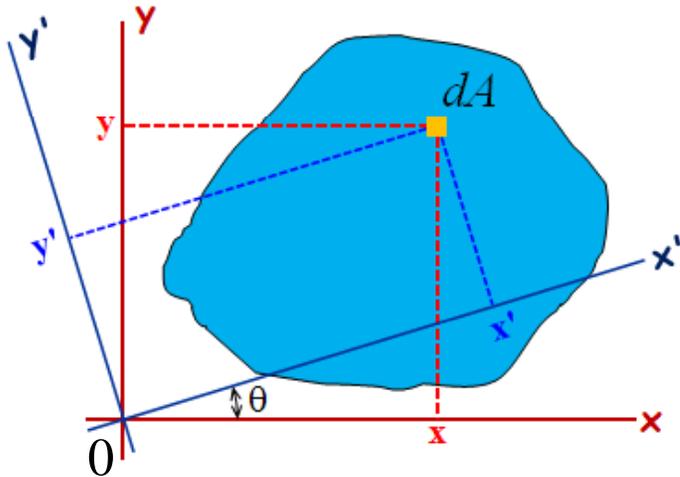
$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta$$

$$I_{y'} = I_x \sin^2 \theta + I_y \cos^2 \theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$



Eixos e Momentos de Inércia Principais



$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta$$

$$I_{y'} = I_x \sin^2 \theta + I_y \cos^2 \theta + I_{xy} \sin 2\theta$$

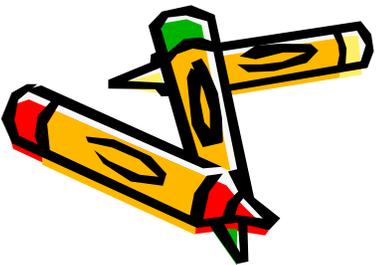
$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

A soma dos momentos de inércia independe do ângulo de giro do sistema de referência, ou seja,

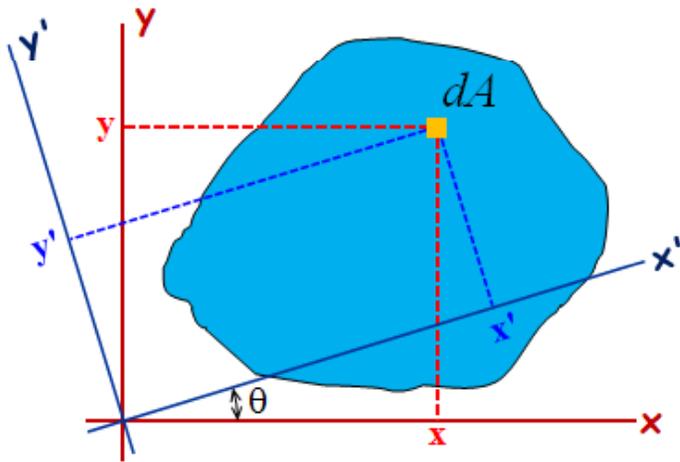
$$I_{x'} + I_{y'} = I_x + I_y = J_0$$

Isso é fato pois a soma dos momentos de inércia leva ao momento polar de inércia, que depende apenas do ponto referente à origem do sistema de referência, que não foi modificado.

Vamos fazer uso dessa identidade para estabelecer $I_{y'}$ sem fazer uso da expressão que depende do ângulo θ .



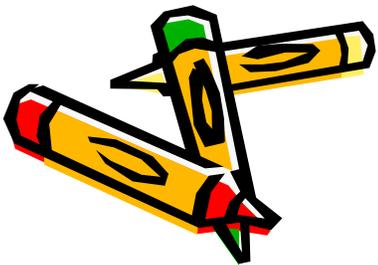
Eixos e Momentos de Inércia Principais



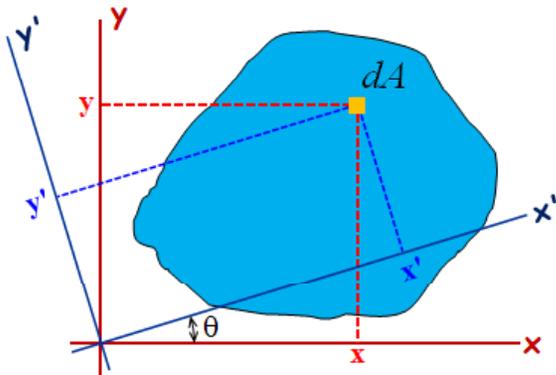
$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

As equações de $I_{x'}$ e $I_{x'y'}$ definem parametricamente uma circunferência para um sistema de coordenadas retangulares com $I_{x'}$ de abscissa e $I_{x'y'}$ de ordenada, para um valor dado de θ .

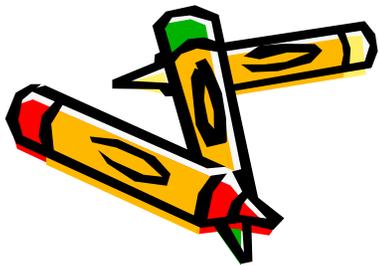
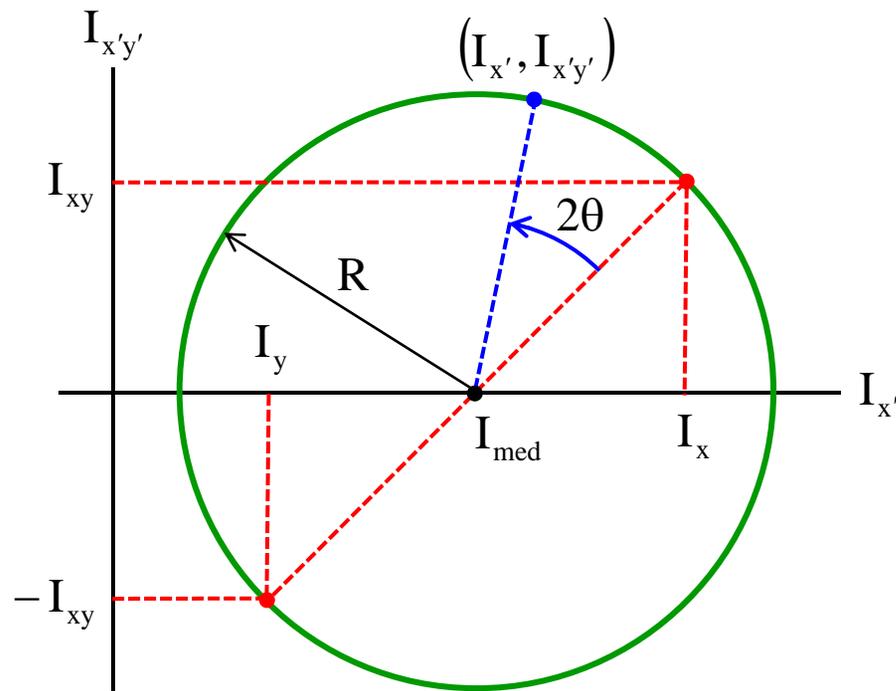


Eixos e Momentos de Inércia Principais

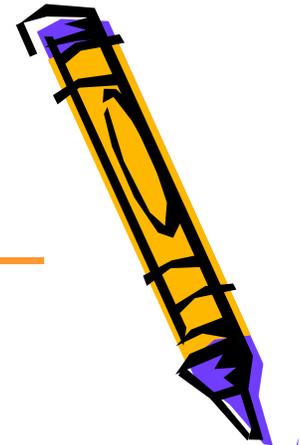


$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta$$

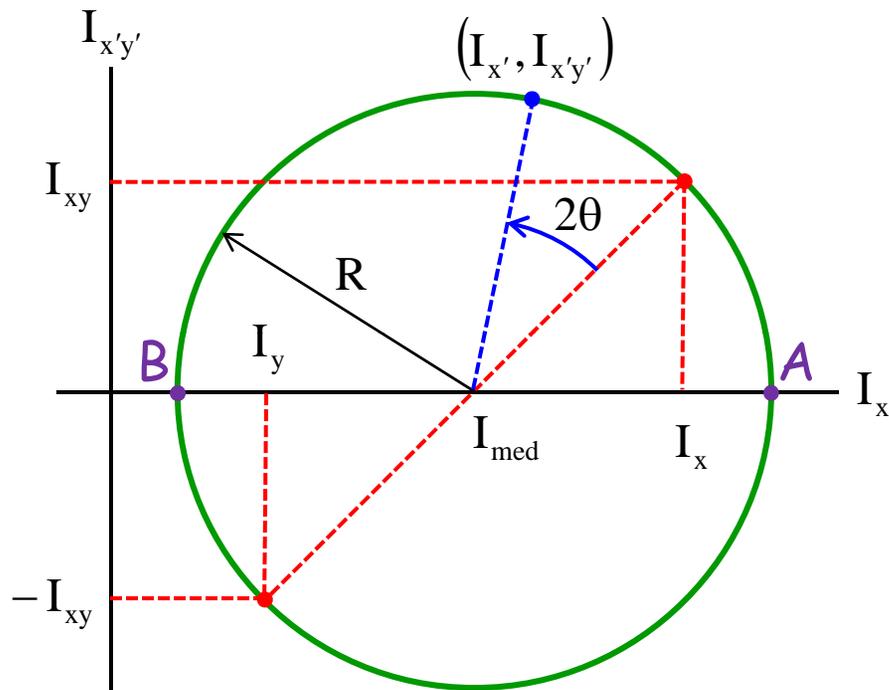
$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$



$$I_{\text{med}} = \frac{I_x + I_y}{2} = \frac{J_0}{2} \quad e \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$



Eixos e Momentos de Inércia Principais

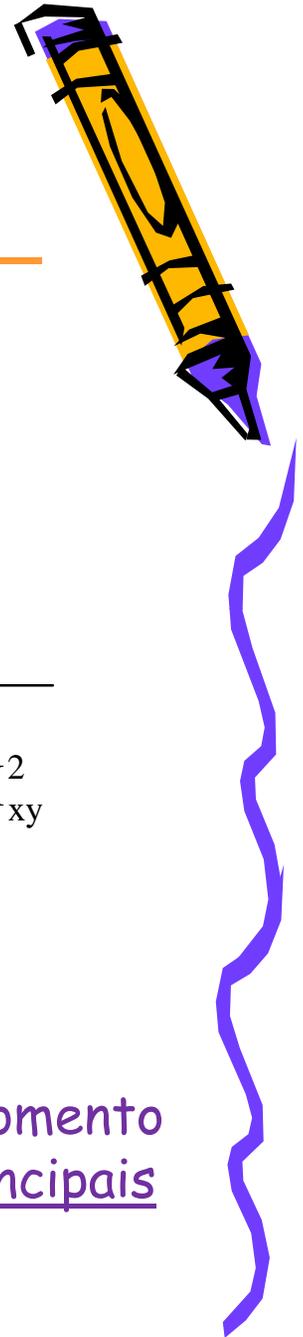
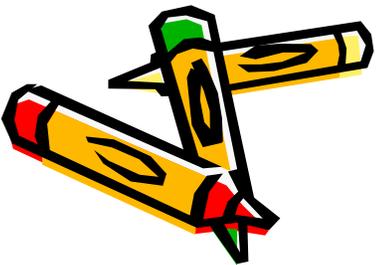


$$I_{\text{med}} = \frac{I_x + I_y}{2} = \frac{J_0}{2}$$

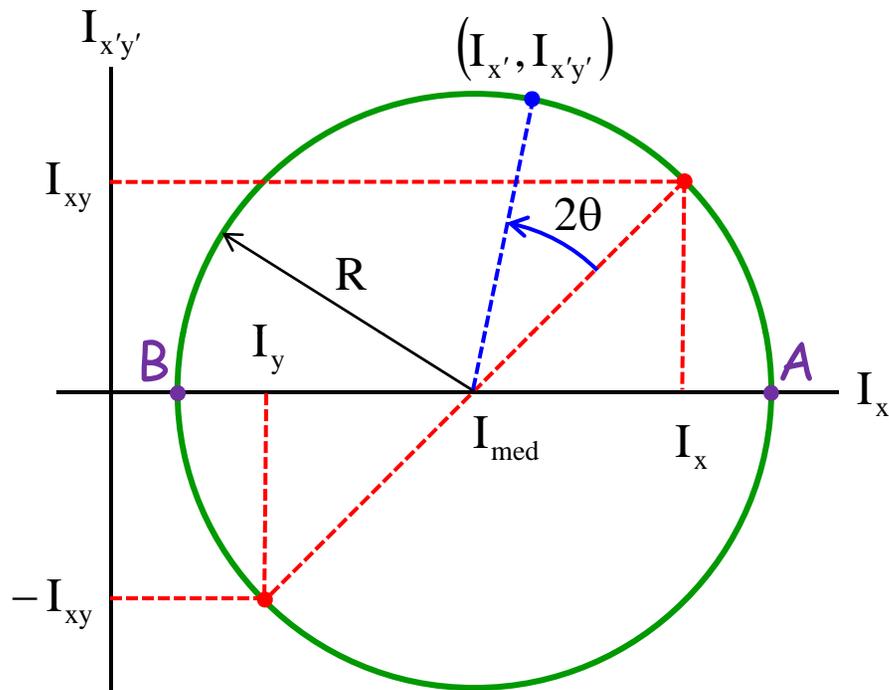
$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

Os pontos A e B são os que apresentam, respectivamente, o maior e o menor valor do momento de inércia, também denominados momentos principais de inércia, dados por

$$I_{\text{max}} = I_{\text{med}} + R \quad e \quad I_{\text{min}} = I_{\text{med}} - R$$



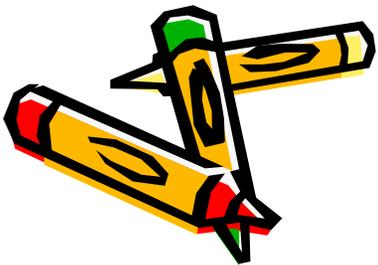
Eixos e Momentos de Inércia Principais



$$I_{\text{med}} = \frac{I_x + I_y}{2} = \frac{J_0}{2}$$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

Ainda em relação aos pontos A e B, o produto de inércia é nulo, o que permite determinar as orientações dos eixos principais de inércia

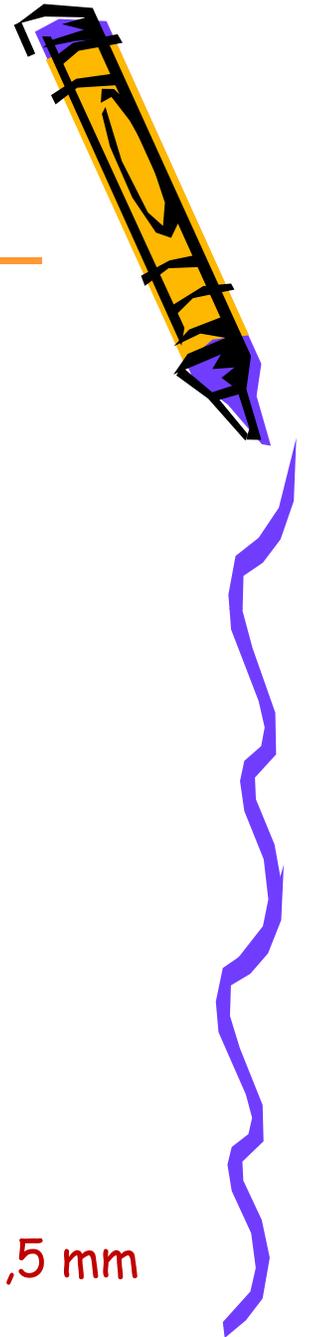
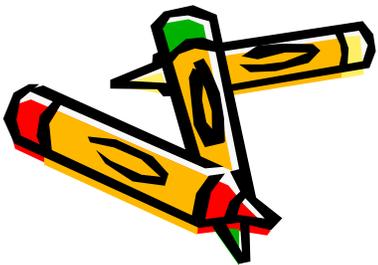
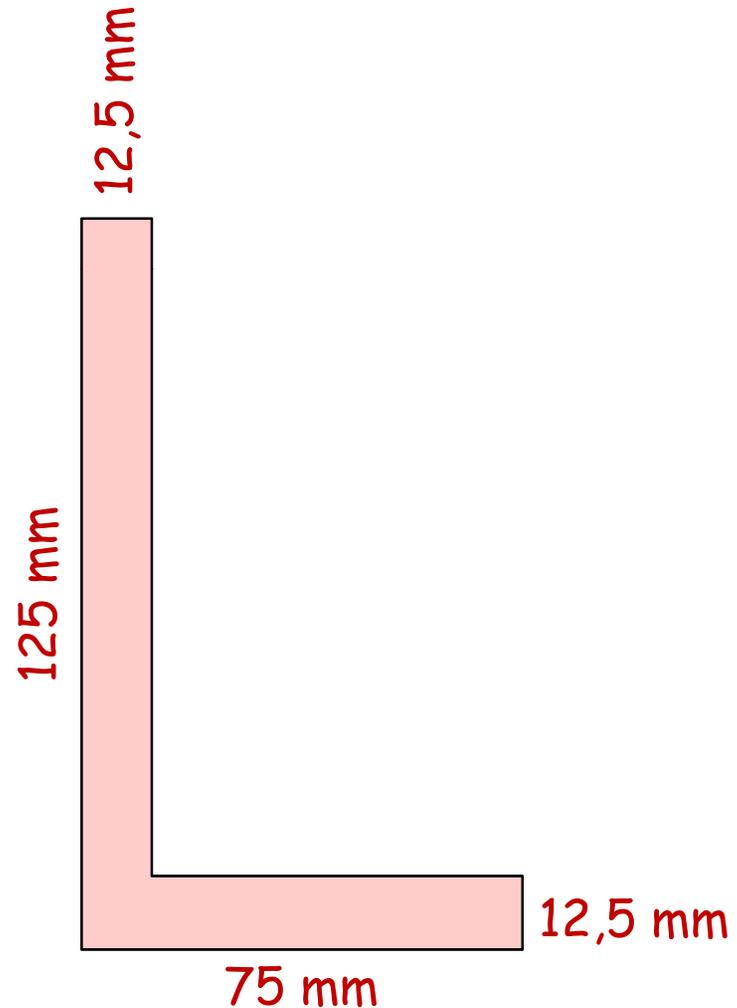


$$I_{x'y'} = 0 \quad \therefore \quad \frac{I_x - I_y}{2} \sin 2\theta_m + I_{xy} \cos 2\theta_m = 0 \quad \Rightarrow \quad \tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

Eixos e Momentos de Inércia Principais

Exemplo:

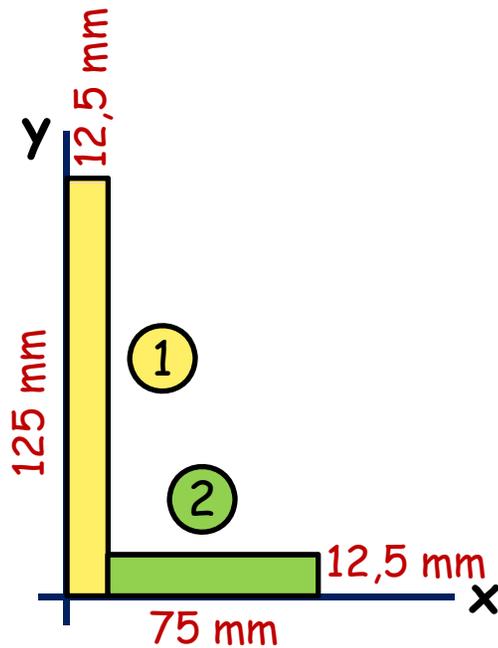
Para a cantoneira em L mostrada, determine a orientação dos eixos centroidais e principais de inércia, bem como os respectivos valores do momento de inércia.



Eixos e Momentos de Inércia Principais



Exemplo (continuação):



$$\bullet (1562,5 + 781,25)\bar{x} = 9765,63 + 34179,69$$
$$\bar{x} = 18,75 \text{ mm}$$

$$\bullet (1562,5 + 781,25)\bar{y} = 97656,25 + 4882,81$$
$$\bar{y} = 43,75 \text{ mm}$$

$$\bullet I_{\bar{x}} = 2583821,61 + 1108805,34$$

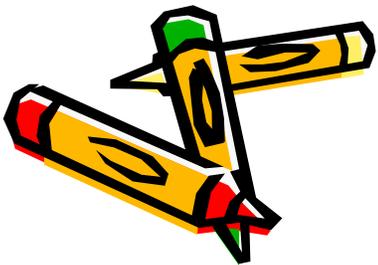
$$I_{\bar{x}} = 3692626,95 \text{ mm}^4$$

$$\bullet I_{\bar{y}} = 264485,68 + 742594,40$$

$$I_{\bar{y}} = 1007080,08 \text{ mm}^4$$

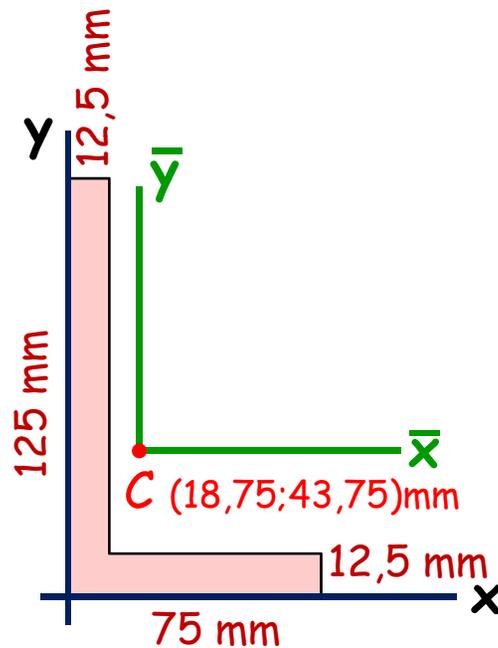
$$\bullet I_{\bar{x}\bar{y}} = -366210,94 - 732421,88$$

$$I_{\bar{x}\bar{y}} = -1098632,81 \text{ mm}^4$$

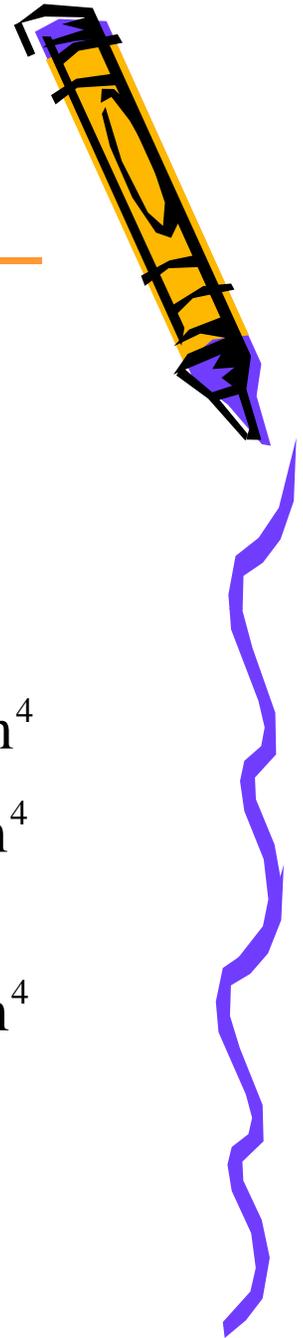
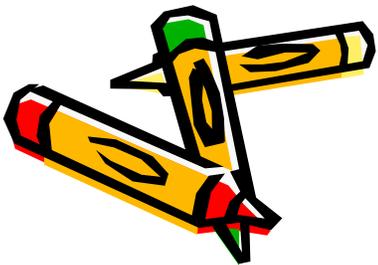


Eixos e Momentos de Inércia Principais

Exemplo (continuação):

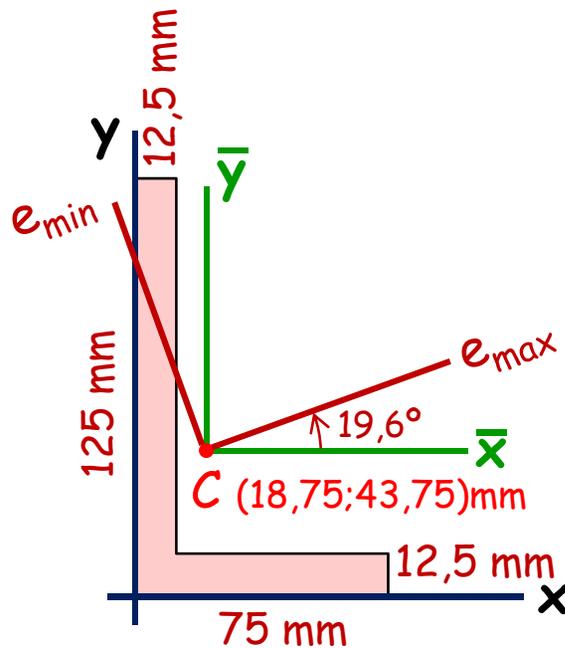


- $I_{\bar{x}} = 3692626,95 \text{ mm}^4$
- $I_{\bar{y}} = 1007080,08 \text{ mm}^4$
- $I_{\bar{x}\bar{y}} = -1098632,81 \text{ mm}^4$
- $I_{\text{med}} = 2349853,52 \text{ mm}^4$
- $R = 1734945,12 \text{ mm}^4$
- $I_{\text{max}} = 4084798,63 \text{ mm}^4$
- $I_{\text{min}} = 614908,40 \text{ mm}^4$
- $\theta_{\text{max}} = 19,6^\circ$
- $\theta_{\text{min}} = 109,6^\circ$



Eixos e Momentos de Inércia Principais

Exemplo (continuação):



- $I_{\bar{x}} = 3692626,95 \text{ mm}^4$
- $I_{\bar{y}} = 1007080,08 \text{ mm}^4$
- $I_{\bar{x}\bar{y}} = -1098632,81 \text{ mm}^4$
- $I_{\max} = 4084798,63 \text{ mm}^4$
- $I_{\min} = 614908,40 \text{ mm}^4$
- $\theta_{\max} = 19,6^\circ$
- $\theta_{\min} = 109,6^\circ$

